Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2016

Problem Set # 7 (due at the beginning of class on November 4)

Notation: If G is a group then the commutator subgroup $G' = [G, G]$ is the subgroup generated by all commutators $xyx^{-1}y^{-1}$ for $x, y \in G$. Then $[G, G] \trianglelefteq G$ and we call $G^{\mathrm{ab}} = G/[G, G]$ is the **abelianization** of G. In fact, G^{ab} is abelian and the canonical projection $G \to G^{\mathrm{ab}}$ is a surjective homomorphism often also called the abelianization.

Let $H \trianglelefteq G$ and $\pi : G \to G/H$ be the natural projection. The following is known as the **universal property of the quotient**: if $\phi : G \to K$ is a group homomorphism such that $H \subset \ker(\phi)$ then there exists a unique homomorphism $F : G/H \to K$ such that $F \circ \pi = \phi$.

Reading: DF 5.1, 5.2, and the beginning of 5.4.

Problems:

1. DF 5.1 Exercises 4, 14, 17* (see 15 for notation).

2. DF 5.2 Exercises 2, 3, 5, 6, 7, 8*, 9, 11, 14*. The notions of **exponent**, **rank**, and **free rank** are defined just above the exercise section.

3. DF 5.4 Exercises 4, 5*, 7, 11, 19*.

4. *Characteristic subgroups.* A subgroup $K \leqslant G$ is called **characteristic** if $\varphi(K) \subset K$ for every $\varphi \in \mathrm{Aut}(G)$. In particular, any characteristic subgroup is normal. For example, the commutator subgroup $[G, G]$ of G is characteristic. Prove the following:

(a) If $H \leqslant G$ is normal and $K \leqslant H$ is characteristic, then $K \leqslant G$ is normal.

(b) If $H \leqslant G$ is characteristic and $K \leqslant H$ is characteristic, then $K \leqslant G$ is characteristic.

(c) The Klein 4-subgroup of S_4 consisting of the $(2, 2)$-cycles and the identity is characteristic. No nontrivial subgroup of this Klein 4-group is characteristic.

(d) The center $Z(G) \leqslant G$ is characteristic.

(e) If G is cyclic, then every subgroup of G is characteristic.

(f) For any fixed n and any group G, the intersection of all subgroups of index n in G is characteristic.

(g) If $H \leqslant G$ is characteristic, then show that the map $\mathrm{Aut}(G) \to \mathrm{Aut}(G/H)$ given by $\varphi \mapsto \overline{\varphi}$ where $\overline{\varphi}(gH) = \varphi(g)H$ is a homomorphism. Give an example where this homomorphism is not injective and an example where this homomorphism is not surjective.

5. *Abelianizing.* Prove the following:

(a) The **universal property of abelianization**: Let $\phi : G \to G^{\mathrm{ab}}$ be the abelianization. For any abelian group H and any homomorphism $f : G \to H$ there exists a unique homomorphism $F : G^{\mathrm{ab}} \to H$ such that $f = F \circ \phi$. Note. This is proved in DF 5.4, but the proof there is a bit long-winded. Use the universal property of the quotient instead.

(b) For any groups H and K, there is an isomorphism $(H \times K)^{\mathrm{ab}} \cong H^{\mathrm{ab}} \times K^{\mathrm{ab}}$. Hint. Use the universal property.

(c) Prove that S_4 is not isomorphic to the direct product $H \times K$ of nontrivial groups. Hint: Abelianize both sides.