Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2016

Problem Set # 4 (due at the beginning of class on Friday 7 October)

Notation: The **sign** $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ of a permutation $\sigma \in S_n$ is defined by $\sigma(\Delta) = \mathrm{sgn}(\sigma)\Delta$, where $\Delta = \prod_{i<j}(x_i - x_j)$. Equivalently, $\mathrm{sgn}(\sigma)$ is 1 (resp. -1) if σ can be written as a product of an even (resp. odd) number of transpositions. The map $\mathrm{sgn} : S_n \to \{\pm 1\}$ is a homomorphism with kernel the alternating group A_n.

Reading: DF 3.3, 3.5.

Problems:

1. DF 3.3 Exercises 2, 3, 6*, 7* (note that in particular, if also $M \cap N = \{e\}$ then $G \cong M \times N$ and we say that G is the **internal direct product** of M and N), 8, 9*.

2. DF 3.5 Exercises 3*, 4*, 6*, 13*, 14, 15, 17*.

3. *More classification.* Prove that if G is an abelian group of order pq, where p an q are distinct primes numbers, then G is cyclic. You can use Cauchy's theorem for abelian groups.

4. *Some isomorphisms.*

(a) For any field F, prove that the center of $\mathrm{GL}_2(F)$ consists of $F^\times$ multiples of the identity matrix. What is the center of $\mathrm{SL}_2(F)$? We denote by $\mathrm{PGL}_2(F) = \mathrm{GL}_2(F)/Z(\mathrm{GL}_2(F))$ and $\mathrm{PSL}_2(F) = \mathrm{SL}_2(F)/Z(\mathrm{SL}_2(F))$.

(b) Prove that $\mathrm{GL}_2(F)$ acts on the set P of lines in F^2 through the origin and that the kernel of this action is the center of $\mathrm{GL}_2(F)$. Here, "line through the origin" is a colloquial term for "1-dimensional subspace." Conclude that $\mathrm{PGL}_2(F)$ acts faithfully on the set P, hence the permutation representation is an injective homomorphism $\mathrm{PGL}_2(F) \to S_P$ to the symmetric group on the elements of P.

(c) Calculate $|\mathrm{PGL}_2(\mathbb{F}_p)|$.

(d) Prove that $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$. (Hint: How many lines through the origin are there in $\mathbb{F}_3^2$?)

(e) For an odd prime p, prove that the map $\mathrm{PSL}_2(\mathbb{F}_p) \to \mathrm{PGL}_2(\mathbb{F}_p)$, taking the coset represented by M to the coset represented by M, is a well defined injective homomorphism whose image has index 2. Notice that for $p = 3$ this is particularly clear!

(f) Prove that $\mathrm{PSL}_2(\mathbb{F}_3) \cong A_4$. Hint: First show that the determinant is a well defined homomorphism $\det : \mathrm{PGL}_2(\mathbb{F}_3) \to \mathbb{F}_3^\times$, then show that under your isomorphism from part (d) the determinant is the same (in the group $\mathbb{F}_3^\times \cong \{\pm 1\}$) as the sign of the corresponding permutation. For this, think of what the transpositions in S_4 look like in $\mathrm{PGL}_2(\mathbb{F}_3)$.

(g) We know that A_4 has a normal subgroup isomorphic to the Klein four group V_4. Via the isomorphism in (f), write the corresponding subgroup in $\mathrm{PSL}_2(\mathbb{F}_3)$ as a group of matrices (modulo ± 1).

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