Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2016

Thanksgiving Problem Set # 10 (due at the beginning of class on Friday 2 December)

Notation: Let R be a ring and $r \in R$. The **principal ideal** $(r) \subset R$ generated by r is the set of all finite sums of elements of the form arb for $a, b \in R$. Verify for yourself that if R is commutative then $(r) = \{ar \mid a \in R\}$.

Let R be a commutative ring with $1 \neq 0$. We will write $R[x_1, x_2, \ldots, x_n]$ for the ring of multivariable polynomials in the variables $x_1, x_2, \ldots, x_n$ and with coefficients in R.

Reading: DF 7.2–7.4.

Problems:

1. DF 7.3 Exercises 1, 10, 14, 17*, 20, 21* (in particular, if F is a field, find all two-sided ideals of $M_n(F)$), 24* (forget the second part of (a)), 26*, 28, 29*, 31, 33, 34.

2. DF 7.4 Exercises 8, 14*, 30, 32*.

3. *Symmetric polynomials.* Let R be a commutative ring with $1 \neq 0$.

(a) Consider the symmetric group S_n acting on the set $\{x_1, \ldots, x_n\}$ by permutations. As usual, extend this action to $R[x_1, x_2, \ldots, x_n]$. For example, if $\sigma = (123) \in S_3$, then

$$\sigma \cdot (x_1x_2 - 2x_3^2 + 3x_2x_3^2) = x_2x_3 - 2x_1^2 + 3x_3x_1^2.$$

Prove that S_n acts on $R[x_1, \ldots, x_n]$ by ring homomorphisms. Hint. Consider monomials.

(b) Let $S \subset R[x_1, \ldots, x_n]$ be the set of multivariable polynomials that are fixed under the action of S_n. Prove that S is a subring with 1. This is called the **ring of symmetric polynomials**.

(c) For each $n \geq 0$, define polynomials $e_i \in R[x_1, \ldots, x_n]$ by $e_0 = 1$ and

$$e_1 = x_1 + \cdots + x_n, \quad e_2 = \sum_{1 \leq i < j \leq n} x_ix_j, \quad \ldots, \quad e_n = x_1 \cdots x_n$$

and $e_k = 0$ for $k > n$. In words, e_k is the sum of all distinct products of subsets of k distinct variables. Prove that each e_k is a symmetric polynomial. These are called the **elementary symmetric polynomials**.

(d) The **generic polynomial** of degree n is the polynomial

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

in the ring $R[x_1, \ldots, x_n][x]$ of polynomials in x with coefficients in $R[x_1, \ldots, x_n]$. Prove (by induction) that

$$f(x) = (x-x_1)(x-x_2) \cdots (x-x_n) = x^n - e_1x^{n-1} + e_2x^{n-2} + \cdots + (-1)^n e_n = \sum_{j=0}^{n} (-1)^{n-j} e_{n-j} x^j.$$

(e) For each $k \geq 1$, define the **power sums** $p_k = x_1^k + \cdots + x_n^k$ in $R[x_1, \ldots, x_n]$. Clearly, the power sums are symmetric. Verify the following identities by hand:

$$p_1 = e_1, \quad p_2 = e_1p_1 - 2e_2, \quad p_3 = e_1p_2 - e_2p_1 + 3e_3$$

In general **Newton's identities** in $R[x_1, \ldots, x_n]$ are (recall that $e_k = 0$ for $k > n$):

$$p_k - e_1p_{k-1} + e_2p_{k-2} - \cdots + (-1)^{k-1}e_{k-1}p_1 + (-1)^k ke_k = 0.$$

Prove Newton's identites whenever $k \geq n$.
Hint. For each i, consider the equation in part (d) for $f(x_i)$ and sum all these equations together. This gives Newton's identity for $k = n$. Set extra variables to zero to get the identities for $k > n$ from this. (Fun. Can you come up with a proof when $1 \leq k \leq n$?)