Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2015

Midterm Exam Review Sheet

Directions: The midterm exam will take place in class on Monday, October 19th. You will have the entire class period, 50 minutes, to complete the exam. No electronic devices will be allowed. No notes will be allowed. On all problems, you will need to write your thoughts/proofs in a coherent way to get full credit.

Topics covered and practice problems:

- Basic set theory and functions (injections, surjections, bijections). Definition of a group. Modular arithmetic. DF 0.1 Exercises 4–6; DF 0.2 Exercises 7, 8, 11; DF 0.3 Exercises 4, 5, 7, 9; DF 1.1 Exercises 1, 2, 5–9, 12–14, 21, 28, 31, 36.

- Dihedral groups. Symmetric groups. Disjoint cycle decomposition of permutations. Matrix groups over a field. Alternating groups. DF 1.2 Exercises 4–6, 9; DF 1.3 Exercises 9–19; DF 1.4 Exercises 7, 10; DF 3.5 Exercises 2–6, 9–12, 15–16.

- Homomorphisms and isomorphisms. Kernel. Image. DF 1.6 Exercises 3–9, 11, 15–16, 19, 21–22, 25.

- Group actions. Permutation representation. Kernel. Faithful. Transitive. Orbit. Stabilizer. Left multiplication action. Conjugation action. Class equation. DF 1.7 Exercises 1–3, 5–6, 8–13, 20, 21, 23; DF 4.1 Exercises 1–6; DF 4.2 Exercises 1–6, 10, 13; DF 4.3 Exercises 2–3, 6–12, 25, 28–29.

- Subgroups. Centralizers. Normalizers. Cyclic subgroups. Generators. Lattice of subgroups. DF 2.1 Exercises 1–5, 14; DF 2.2 Exercises 1–2, 7, 10–11; DF 2.3 Exercises 1–5, 9, 11–13, 15; DF 2.4 Exercises 6–9, 18; DF Exercises 4, 6–10;

- Quotient groups. Cosets. Isomorphism theorems. Composition series. Simple groups. Solvable groups. DF 3.1 Exercises 6–13, 31, 33–34; DF 3.2 Exercises 4, 8, 13–16, 22–23; DF 3.3 Exercises 1, 3, 8; DF 3.4 Exercises 1–2, 5, 7.

- Automorphism groups. Inner automorphisms. DF 4.4 Exercises 1, 3, 5, 12–14, 15, 16.

- Sylow p-subgroups. DF 4.5 Exercises 5–9.

Practice exam questions:

1. There will be several True/False problems covering a range of topics so far: isomorphism classes of groups, orders of elements, Lagrange's theorem, Cauchy's theorem, group actions, and Sylow's theorem.

2. Let $V_1 \subset \mathbb{R}^2$ be the subset of all vectors whose slope is an integer. Let $V_2 \subset \mathbb{R}^2$ be the subset of all vectors whose slope is a rational number. Determine if V_1 and/or V_2 is a subgroup of $\mathbb{R}^2$, with usual vector addition.

3. Write down a nontrivial homomorphism $\mathbb{Z}/36\mathbb{Z} \to \mathbb{Z}/48\mathbb{Z}$ and compute its image and kernel.

4. How many elements of order 6 are there in S_6? In A_6?

5. Prove that $11^{104} + 1$ is divisible by 17.

6. Let G be a finite group and $x, y \in G$ be nonindentity elements such that $xyx^{-1} = y^{-1}$. Show that G has an element of order 2. Find an infinite group G where this conclusion fails.

7. Write down two elements of S_{10} that generate a subgroup isomorphic to D_{10}. (Hint: Use the left multiplication action on D_{10}.)

8. Consider the permutation representation $S_n \to S_{n!}$. Describe the cycle type in $S_{n!}$ of the image of an n-cycle in S_n.

9. Prove that $C_{S_n}\big((12)(34)\big)$ has $8\,(n-4)!$ elements for $n \geq 4$ and explicitly determine all of them.

10. Show that the set of nonzero matrices of the form

$$\begin{pmatrix} a & 3b \\ b & a \end{pmatrix}$$

is a cyclic subgroup of $\mathrm{GL}_2(\mathbb{F}_5)$. What is the order of this subgroup?

11. Find square matrices A and B with coefficients in the field $\mathbb{F}_p$ such that $AB - BA = I$. What size do the matrices need to be? (Note that this is impossible for real matrices because $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$!)

12. Find the highest power of p dividing the order of $\mathrm{GL}_n(\mathbb{F}_p)$. Find a Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$. (Hint: Think upper triangular.)

Yale University, Department of Mathematics, 10 Hillhouse Ave, New Haven, CT 06511
E-mail address: asher.auel@yale.edu