Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2015

Problem Set # 5 (due at the beginning of class on Friday 30 October)

Reading: DF 4.1–4.5.

Problems: (Starred* problems are strongly recommended!)

1. DF 4.1 Exercises 3*, 4, 6, 7*, 8, 9.

2. DF 4.2 Exercises 2, 10*, 11, 13*, 14.

3. DF 4.3 Exercises 5*, 8, 9, 13, 19*, 20, 21, 22, 24, 25, 28*, 29, 34.

4. DF 4.4 Exercises 1, 3, 5*, 10, 11, 18*, 19.

5. DF 4.5 Exercises 8*, 16, 18, 22, 26, 30, 39, 40*.

6. *Finite vector spaces**. Let V be an $\mathbb{F}_p$-vector space of (finite) dimension n.

(a) What is the isomorphism type of the underlying (finite) abelian group $(V, +)$?

(b) Show that the automorphism group $\mathrm{Aut}((V, +))$ of the abelian group $(V, +)$ is isomorphic to the group $\mathrm{GL}(V)$ of $\mathbb{F}_p$-linear vector space isomorphisms $\varphi : V \to V$ and that this group is also isomorphic to $\mathrm{GL}_n(\mathbb{F}_p)$.

(c) Compute the order of $\mathrm{Aut}(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$. Find an automorphism of order 7.

7. *Characteristic subgroups**. A subgroup $K \leqslant G$ is called **characteristic** if $\varphi(K) \subset K$ for every $\varphi \in \mathrm{Aut}(G)$. In particular, any characteristic subgroup is normal. Prove the following.

(a) If $H \leqslant G$ is normal and $K \leqslant H$ is characteristic, then $K \leqslant G$ is normal.

(b) If $H \leqslant G$ is characteristic and $K \leqslant H$ is characteristic, then $K \leqslant G$ is characteristic.

(c) The Klein 4-subgroup of S_4 is characteristic. No nontrivial subgroup of the Klein 4-group is characteristic.

(d) For any group G, the commutator subgroup $[G, G]$, generated by the commutators $xyx^{-1}y^{-1}$ for all $x, y \in G$, is characteristic.

(e) The center $Z(G) \leqslant G$ is characteristic.

(f) If G is cyclic, then every subgroup of G is characteristic.

(g) For any fixed n and any group G, the intersection of all subgroups of index n in G is characteristic.

(h) If $H \leqslant G$ is characteristic, then there is a natural homomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(G/H)$.

Yale University, Department of Mathematics, 10 Hillhouse Ave, New Haven, CT 06511
E-mail address: asher.auel@yale.edu