Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2015

Problem Set # 4 (due at the beginning of class on Friday 9 October)

Notation: A group G is **solvable** if there exists a chain of subgroups

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{r-1} \trianglelefteq G_r = G$$

with each G_{i+1}/G_i abelian for all $0 \le i \le r-1$.

Reading: DF 3.1–3.5.

Problems: (Starred* problems are strongly recommended!)

1. DF 3.2 Exercises 5, 17.

2. DF 3.3 Exercises 3, 6*, 8, 9*.

3. DF 3.4 Exercises 2, 7, 8*.

4. DF 3.5 Exercises 3*, 4, 6, 10, 13*, 14, 15, 17*.

5. *More classification*.* Prove that if G is an abelian group of order pq, where p an q are distinct primes numbers, then G is cyclic. (Hint: Use Cauchy's theorem for abelian groups.)

6. *Some isomorphisms*.*

(a) For any field F, prove that the center of $\mathrm{GL}_2(F)$ consists of $F^\times$ multiples of the identity matrix. What is the center of $\mathrm{SL}_2(F)$? We denote by $\mathrm{PGL}_2(F) = \mathrm{GL}_2(F)/Z(\mathrm{GL}_2(F))$ and $\mathrm{PSL}_2(F) = \mathrm{SL}_2(F)/Z(\mathrm{SL}_2(F))$.

(b) Prove that $\mathrm{GL}_2(F)$ acts on the set P of lines in F^2 through the origin and that the kernel of this action is the center of $\mathrm{GL}_2(F)$. Here, "line through the origin" is a colloquial term for "1-dimensional subspace." Conclude that $\mathrm{PGL}_2(F)$ acts faithfull on the set P, hence the permutation representation is an injective homomorphism $\mathrm{PGL}_2(F) \to S_P$ to the symmetric group on the elements of P.

(c) Calculate $|\mathrm{PGL}_2(\mathbb{F}_p)|$.

(d) Prove that $\mathrm{PGL}_2(\mathbb{F}_3) \cong S_4$. (Hint: How many lines through the origin are there in $\mathbb{F}_3^2$?)

(e) For an odd prime p, prove that the map $\mathrm{PSL}_2(\mathbb{F}_p) \to \mathrm{PGL}_2(\mathbb{F}_p)$, taking the coset represented by M to the coset represented by M, is a well defined injective homomorphism whose image has index 2. Notice that for $p = 3$ this is particularly clear!

(f) Conclude that $\mathrm{PSL}_2(\mathbb{F}_3) \cong A_4$. You may do this in two ways. The cheap way is to appeal, without proof, to a statement that you will prove on the next problem set: $A_n \leqslant S_n$ is the unique subgroup of index 2. The fun way is as follows: first show that the determinant is a well defined homomorphism $\det : \mathrm{PGL}_2(\mathbb{F}_3) \to \mathbb{F}_3^\times$, then show that under your isomorphism from part (d) the determinant is the same (in the group $\mathbb{F}_3^\times \cong \{\pm 1\}$) as the sign of the corresponding permutation. Hint: Think of what the 2-cycles in S_4 look like in $\mathrm{PGL}_2(\mathbb{F}_3)$.

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