Yale University Department of Mathematics
Math 350 Introduction to Abstract Algebra
Fall 2015

Problem Set # 10 (due at the beginning of class on Friday 11 December)

Reading: DF 7.4–7.6, 8.1–8.3, 9.1–9.2.

Problems: (Starred* problems are strongly recommended!)

1. DF 7.4 Exercises 37*, 38, 39*.

2. DF 7.5 Exercises 3, 5.

3. DF 7.6 Exercise 5a.

4. DF 8.1 Exercises 3, 6*, 12*.

5. DF 8.2 Exercises 3, 5.

6. DF 8.3 Exercise 8.

7. DF 9.1 Exercises 6, 13* (**Hint.** For any commutative ring R with 1 and any $g \in R$, prove that $R[x]/(x-g) \cong R$, then use this to prove that $y^2 - x$ is prime in $F[x,y]$).

8. DF 9.2 Exercises 1, 2*, 3* (this provides a way to build more finite fields).

9. *Finite field with p^2 elements.* Before, we constructed $\mathbb{F}_4 = \mathbb{F}_2[x]/(x^2+x+1)$. In an analogous way, construct $\mathbb{F}_9$, $\mathbb{F}_{25}$, and $\mathbb{F}_{49}$.

10. *Parabola*.* Let F be a field.

(a) Prove that for any $a_1, \ldots, a_n \in F$, the ideal $(x_1 - a_1, \ldots, x_n - a_n) \subset F[x_1, \ldots, x_n]$ is maximal.
Hint. Consider evaluating a polynomial at $(a_1, \ldots, a_n)$. Then you can proceed as follows: by considering the automorphism $f(x_1, \ldots, x_n) \mapsto f(x_1+a_1, \ldots, x_n+a_n)$ of $F[x_1, \ldots, x_n]$ you can reduce to the case where the ideal is $(x_1, \ldots, x_n)$, which is easier.

(b) Prove that every maximal ideal $M \subset F[x_1, \ldots, x_n]$, such that there is an F-algebra isomorphism $F[x_1, \ldots, x_n]/M \cong F$, is of the form $M = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in F$.
Hint. Given a surjective ring homomorphism $F[x_1, \ldots, x_n] \to F$ with kernel M, consider the images of x_i.

(c) Show that (x^2+1, y) is a maximal ideal in $\mathbb{R}[x,y]$ and compute its quotient. Note that this maximal ideal is not in the form as in the previous parts.

(d) Consider the ideal $I = (y - x^2)$ in $\mathbb{R}[x,y]$. Show that the maximal ideals $M \subset \mathbb{R}[x,y]$ with $\mathbb{R}[x,y]/M \cong \mathbb{R}$ and $I \subset M$ are exactly those of the form $M = (x-a, y-b)$ for $(a,b) \in \mathbb{R}^2$ on the parabola $y = x^2$. Think about how you might characterize the maximal ideals $M \subset \mathbb{R}[x,y]$ containing I such that $\mathbb{R}[x,y]/M \cong \mathbb{C}$.