Math 235 Reflection Groups

Review of theorems and definitions for the final exam

(1) Definition of a rotation, reflection, orthogonal transformation in a Euclidean vector space.

(2) Thm 2.2.1 - Classification of finite groups of orthogonal transformations in 2 dimensions.

(3) Thms 2.3.1 and 2.3.2 - Geometric effect of a rotation and of a transformation with determinant -1 in $\mathbb{R}^3$.

(4) Effect of negating a rotation.

(5) Definition of G^* and $G]H$ for the rotation groups G and $H \subset G$.

(6) Thm 2.5.2 - Classification of finite groups of orthogonal transformations in 3 dimensions.

(7) Definition of a fundamental region for a finite group of orthogonal transformations. (Ch. 3)

(8) Prop 3.1.1 - A vector space of dimension ≥ 1 is not a union of a finite number of its proper subspaces.

(9) Definition of a Coxeter group.

(10) Definition of a root of a Coxeter group; the root system and simple roots.

(11) Prop 4.1.1 - A Coxeter group G acts on its root system.

(12) Prop 4.1.4 - A root cannot be a linear combination of two simple roots with coefficients of opposite signs.

(13) Prop 4.1.5 - A simple reflection S_i maps a simple root r_j, $i \neq j$, to a positive root. For any two distinct simple roots, the inner product satisfies $(r_i, r_j) \leq 0$.

(14) Prop 4.1.6 - A finite set of vectors lying on one side of a hyperplane and such that their mutual inner products are nonpositive, is linearly independent.

(15) Prop 4.1.7 - The simple roots form a basis in the vector space where the group acts effectively.

(16) Prop 4.1.8 - Uniqueness of the t-base Π_t for a root system.

(17) Prop 4.1.9 - A simple reflection maps positive roots to positive roots, except for its own root.

(18) Prop 4.1.10 - Any vector can be mapped to the fundamental region of G by an element of G.

(19) Prop 4.1.11 - Any roots can be mapped to a simple root by an element of G.

(20) Thm 4.1.12 - The simple reflections generate G.

(21) Prop 4.2.3 - The only element of G that maps all positive roots to positive roots is the identity.

(22) The description of the fundamental region of a Coxeter group G: $F = \{x \in V : (x, r_i) \geq 0, \quad 1 \leq i \leq n\}$.

(23) Thm 4.2.5. - Every reflection in G is conjugate to a fundamental reflection and every root of G is in the root system Δ.

(24) Prop 5.1.1 - For any $r_i, r_j \in \Pi$, there is an integer $p_{ij} \geq 1$ such that the cosine of the angle between r_i and r_j is $-\cos(\pi/p_{ij})$. Moreover, p_{ij} is the order of the element $S_i S_j$ in G.

(25) Definition of the Coxeter graph.

(26) Thm 5.1.2 - If two Coxeter groups have the same Coxeter graph, they are geometrically equivalent.

(27) Thms 5.1.3, 5.1.4 - The Coxeter graph of an irreducible Coxeter group is connected and positive definite.

(28) Prop 5.1.5 - The recursive formula for the determinant of a Coxeter graph.

(29) Prop 5.1.5 - A subgraph of a positive definite graph Γ is also positive definite.

(30) Thm 5.1.7 - Classification of connected positive definite Coxeter graphs.

(31) Thm 5.2.2 - Classification of connected positive definite Coxeter graphs satisfying the crystallographic condition.

(32) Group-theoretic description of the groups $A_n \simeq S_{n+1}$, $B_n \simeq S_n \ltimes K_n$, $D_n \simeq S_n \ltimes L_n$ (Ch. 5.3). (Don't worry so much about the general notion of semi-direct product of groups, just understand what it means in the case of B_n and D_n.)

(33) Thm 5.3.1 - Classification of finite groups of orthogonal transformations generated by reflections.

(34) Thm 5.4.1 - If a subgroup $H \subset G$ leaves invariant a vector orthogonal to all simple roots but r_i, then H is generated by all simple reflections of G but S_i.

(35) Thm 5.4.2 - An irreducible simply laced (i.e., no markings greater than 3) Coxeter group G acts transitively on its root system.