Yale University Department of Mathematics
Math 235 Reflection Groups
Spring 2016
Midterm Exam Problems

Directions: The midterm exam will take place in class on Thursday, March 31. You will have the entire class period, 75 minutes, to complete the exam, which will consist of 2 problems randomly selected from the problems listed below. No electronic devices will be allowed. No notes will be allowed. You will need to write your thoughts/proofs in a coherent way to get full credit.

Reading: GB 1.1–5.1.

Problems:

♠**1** Consider the transformations in $\mathbb{R}^2$ given by the following matrices with respect to a standard orthonormal basis:

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad S_1 = \begin{pmatrix} \cos\pi/3 & \sin\pi/3 \\ \sin\pi/3 & -\cos\pi/3 \end{pmatrix}$$

What group do they generate? Give the name and the order of the group, and find the order of the elements S_0, S_1, and S_0S_1.

♠**2** (a) Suppose G is generated by two reflections S_1 and S_2 in $\mathbb{R}^2$, such that the angle between the axes of the reflections is $12.5°$. What group do they generate? If it is finite, identify this group according to the classification of the finite groups in $O(\mathbb{R}^2)$, and give its order.
(b) What is the smallest subgroup of $O(\mathbb{R}^2)$ that contains rotations by $12°$ and by $45°$?

♠**3** Consider the dihedral group H_2^5 acting as a group of symmetries of a regular pentagon. In particular, this group acts by permutations of the 5 vertices of the pentagon. Identify H_2^5 as a subgroup in the symmetric group S_5. Which permutations of 5 elements does it contain? Is this subgroup normal in S_5?

♠**4** How many conjugacy classes are there in the symmetric group S_5? (Count the number of partitions of 5 elements). How many 3-cycles and 4-cycles are there in S_5? How many products of two disjoint transpositions are there? Explain how you obtained the answers.

♠**5** Suppose that $\dim(V)$ is odd. Let $G \subset O(V)$ be a group of orthogonal transformations in V, and $H \subset G$ is the subgroup of all rotations in G. Suppose that $H \neq G$.
(a) Show that

$$K = H \cup \{-T : T \in G \setminus H\}$$

is a group that consists of rotations. Where did you use that $\dim(V)$ is odd?
(b) Can the set $G \setminus H$ have an element of odd order?

♠**6** Suppose that $\dim(V)$ is odd. Let $G \subset O(V)$ be a group of orthogonal transformations in V, and $H \subset G$ is the subgroup of all rotations in G. Suppose that $H \neq G$. Show that the group

$$K = H \cup \{-T : T \in G \setminus H\}$$

is isomorphic to G if and only if $-1 \notin G$. What is K isomorphic to if $-1 \in G$?

♠**7** Describe geometrically the action of all elements in the groups C_3^4 and H_3^4. For each of the groups, describe a fundamental region.

♠**8** Describe geometrically the action of all elements in the groups $(C_3^5)^*$ and $H_3^5]C_3^5$. For each of the groups, describe a fundamental region.

♣**1** Consider the group of rotational symmetries of a cube W acting by permutations on the set of vertices of the cube. This allows us to see W as a subgroup in S_8. What is the cycle type in S_8 of (a) a rotation by $\pi/2$, (b) a rotation by $2\pi/3$, (c) a rotation by π? Choose an ordering of the vertices and write down an example of each transformation as an element of S_8. Can you identify W as a subgroup in S_6?

♣**2** We know that the group T of rotational symmetries of a tetrahedron is isomorphic to A_4, the alternating group of 4 elements. Set $T^* = T \cup \{-R \,:\, R \in T\}$.
(a) Show that T^* is not isomorphic to the symmetric group S_4.
(b) Show that T^* is not a group of symmetries of a regular tetrahedron.

♣**3** (a) List all reflections in the group $T^* = T \cup \{-R \,:\, R \in T\}$.
(b) Show that T^* is not generated by its reflections.
(c) What subgroup do the reflections in T^* generate? Identify it as one of the groups of orthogonal transformations in $\mathbb{R}^3$.

♣**4** In the root system of $W]T$, with simple roots chosen to be $\{e_1 - e_2,\ e_2 - e_3,\ -e_1 - e_2\}$, give an example of (a) a positive root (but not simple), (b) a negative root, (c) a vector that is neither positive nor negative, (d) a simple weight (an element of the basis dual to the set of the simple roots). Give your answers in terms of linear combinations of the standard basis $\{e_i\}_{i=1}^3$.
(e) Use the root system to find the angles between the lines connecting the middles of the opposite edges in a regular tetrahedron.

♣**5** We said in class that a Euclidean vector space V of dimension $\dim(V) \geq 1$ is not a finite union of its proper subspaces. Use this statement to show that if Y is any finite subset of vectors in V with $0 \notin Y$, then there is a vector $t \in V$ such that $\langle y, t\rangle \neq 0$ for all $y \in Y$.

♣**6** Prove the Cauchy-Schwarz inequality: for any two vectors x, y in a Euclidean space V with the inner product $\langle \cdot, \cdot \rangle$,

$$\langle x, y\rangle^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle,$$

and the equality holds if and only if the vectors x and y are linearly dependent. *Hint:* consider the inner square of a vector $ax + by$ for a suitable choice of coefficients a and b.

♣**7** List all groups of orthogonal transformations in $\mathbb{R}^3$ of order 24. Which of them contain(s) (a) an element of order 6, (b) an element of order 12?

♣**8** Consider the action of S_n in the Euclidean space $\mathbb{R}^n$ by permutations of an orthonormal basis $\{e_1, e_2, \ldots e_n\}$.
(a) Show that S_n acts effectively in the subspace orthogonal to the vector

$$e_1 + e_2 + \ldots + e_n.$$

(b) Find the root system corresponding to this Coxeter group. How many roots does it contain?

♢**1** Describe the subgroup generated by all the reflections in $(H_3^4)^*$. Is $(H_3^4)^*$ a Coxeter group? If so, describe the root system of the group. Otherwise, describe the root system of the subgroup in $(H_3^4)^*$ generated by the reflections.

♢**2** Describe the subgroup generated by all the reflections in $H_3^8]H_3^4$. Is $H_3^8]H_3^4$ a Coxeter group? If so, describe the root system of the group. Otherwise, describe the root system of the subgroup in $H_3^8]H_3^4$ generated by the reflections.

$\diamondsuit$**3** How many reflections are there in $C_3^6]C_3^3$? Is it a Coxeter group? What group do the reflections in $C_3^6]C_3^3$ generate? Describe the root system of this group.

$\diamondsuit$**4** Let Π_t be a t-base (the set of simple roots) for a given Coxeter group acting in a Euclidean space V. Consider the open fundamental region

$$F_t^0 = \{x \in V : (x, r_i) > 0 \text{ for all } r_i \in \Pi_t\}.$$

Show that

$$F_t^0 = \{u \in V : \Pi_u = \Pi_t\}.$$

$\diamondsuit$**5** Suppose that Π is a set of simple roots for a root system Δ. If $r \in \Delta^+$, show that $r \in \Pi$ if and only if r is not a strictly positive linear combination of two or more positive roots.

$\diamondsuit$**6** Consider the action of S_3 by permutations of the standard basis in $\mathbb{R}^3$. Also consider such permutations composed with reflections along the standard basis elements. What isc the root system of the group generated by all these transformations? What is its order? Identify it with one of the group in the classification of the finite orthogonal groups in $\mathbb{R}^3$.

$\diamondsuit$**7** Can the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

be an element of a finite group G generated by reflections in $\mathbb{R}^4$, if it is
(a) written in a basis of simple roots of G,
(b) written in the standard orthonormal basis in $\mathbb{R}^4$?
Explain your answer.

$\diamondsuit$**8** Suppose that a Coxeter group in $\mathbb{R}^3$ has a set of simple roots coinciding with an orthonormal basis $\{e_1, e_2, e_3\}$. What is the group? Find its order and describe its multiplicative structure. Identify this group according to the classification of finite groups of orthogonal transformations in $\mathbb{R}^3$.

$\heartsuit$**1** Let $\{x_i\}_{i=1}^n$ be a basis in a Euclidean space V and let $\{y_i\}_{i=1}^n$ be the dual basis, so that $\langle x_i, y_j \rangle = \delta_{ij}$. Suppose that A is the matrix with ijth entry $\langle x_i, x_j \rangle$ and B the matrix with ijth entry $\langle y_i, y_j \rangle$. Prove that $B = A^{-1}$. *Hint:* What is the decomposition of the vector x_i in the basis $\{y_j\}_{j=1}^n$? What is the decomposition of the vector y_i in the basis $\{x_j\}_{j=1}^n$?

$\heartsuit$**2** Let $\Pi = \{r_i\}_{i=1}^k$ be a set of simple (fundamental) roots for a Coxeter group G, and let $P_i = r_i^\perp$ be the orthogonal hyperplanes for $1 \le i \le k$.
(a) Describe the subspace $P_{\text{all}} = \cap_{i=1}^k P_i$.
(b) Describe the subspace $P_{\neq j} = \cap_{1 \le i \le k, i \neq j} P_i$. *Hint:* this subspace has a nice description in terms of the dual basis Π^*.

$\heartsuit$**3** If V is an n-dimensional Euclidean space and $\{x_1, \dots x_{n+2}\}$ are $n+2$ vectors in V, show that $\langle x_i, x_j \rangle \ge 0$ for some $i \neq j$.

$\heartsuit$**4** Let V be a Euclidean vector space, and Π a set of simple roots for a Coxeter group acting in V. If $x, y \in V$ with $\langle x, r_i \rangle > 0$ and $\langle y, r_i \rangle > 0$ for all $r_i \in \Pi$, show that $\langle x, y \rangle \ge 0$.

$\heartsuit$**5** The action of the symmetric group S_2 in $\mathbb{R}^2$ by permutation of the orthonormal basis elements $\{e_1, e_2\}$ can be composed with the reflections P_1 along e_1 and P_2 along e_2. Show that the obtained group acts effectively in $\mathbb{R}^2$. Describe its root system and sketch its Coxeter graph.

♡**6** Find the simple root system in $\mathbb{R}^2$ corresponding to the Coxeter graphs

$$\underset{8}{\circ - - - - - \circ}$$

and

$$\circ \qquad\qquad \circ$$

Identify each group according to the classification of all finite orthogonal groups in $\mathbb{R}^2$. Find the order of each group. Is one of the groups geometrically equivalent to a subgroup in the other? Justify your answer.

♡**7** Find a set of simple roots in $\mathbb{R}^3$ and identify the Coxeter group corresponding to the Coxeter graphs

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and

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Identify each group according to the classification of all finite orthogonal groups in $\mathbb{R}^3$. Find the order of each group. Justify your answer.

♡**8** Find a set of simple roots in $\mathbb{R}^3$ and identify the Coxeter groups corresponding to the Coxeter graphs

$$\underset{4}{\circ - - - - - \circ} \qquad\qquad \circ$$

and

$$\circ \qquad\qquad \underset{5}{\circ - - - - - \circ}$$

Identify each group according to the classification of all finite orthogonal groups in $\mathbb{R}^3$. Find the order of each group. Justify your answer.