Yale University Department of Mathematics
**Math 235 Reflection Groups**
Spring 2016

Problem Set # 9 (due at the beginning of class on Thursday 28 April)

**Reading:** GB 5.

**Problems:**

**1.** Let $\alpha = \cos(\pi/5)$ and $\beta = \cos(2\pi/5)$.

(a) Verify that $4\alpha^2 = 2\alpha + 1$, $4\beta^2 = -2\beta + 1$, and $2\alpha = 2\beta + 1$.

(b) Verify that the vectors

$$\begin{aligned} r_1 &= (2\alpha+1)\beta e_1 + \beta e_2 - 2\alpha\beta e_3 \\ r_2 &= -(2\alpha+1)\beta e_1 + \beta e_2 + 2\alpha\beta e_3 \\ r_3 &= 2\alpha\beta e_1 - (2\alpha+1)\beta e_2 + \beta e_3 \end{aligned}$$

in $\mathbb{R}^3$ form the base of a root system of type $I_3$.

(c) Give an interpretation of the root system $\Delta$ of type $I_3$ in terms of the geometry of the icosahedron. Sketch the simple roots in relationship to the icosahedron. How must the icosahedron be positioned in $\mathbb{R}^3$? Write all the elements of $\Delta^+$.
**Hint.** For the last part, you can either use the geometry of the icosahedron or the algorithm presented in class for generating $\Delta^+$ from $\Pi$.

(d) Viewing the vectors $r_1, r_2, r_3$ in $\mathbb{R}^4$ and further defining

$$r_4 = -2\alpha\beta e_1 - (2\alpha+1)\beta e_3 + \beta e_4,$$

show that $\{r_1, r_2, r_3, r_4\}$ forms the base of a root system of type $I_4$.

**2.** Let $\Gamma = \{a_1, a_2, \ldots a_n\}$ be the Coxeter graph of an irreducible Coxeter group $G$, and $\{a_1, \ldots a_k\}$ a subset of nodes of $\Gamma$, such that $\Gamma_1 = \Gamma \setminus \{a_1, \ldots a_k\}$ is the Coxeter graph of an irreducible Coxeter group $G_1$. Then $G_1$ is a subgroup of $G$.

(a) Can $G_1$ be normal in $G$?

(b) List all irreducible Coxeter subgroups of $B_4$. Are any of them normal and why?

**3.** Let $\{e_i\}_{i=1}^8$ be an orthonormal basis in $\mathbb{R}^8$. We will construct the 240 roots of $E_8$:

$$\Delta = \left\{ \begin{array}{lll} \pm e_i \pm e_j, & i \neq j, & 1 \leq i,j \leq 8 \\ \frac{1}{2}\sum_{i=1}^8 \varepsilon_i e_i, & \varepsilon_i = \pm 1, & \prod_{i=1}^8 \varepsilon_i = -1 \end{array} \right\}$$

The simple roots of $E_8$ are

$$\Pi = \{r_1 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7 - e_8),\ \ r_2 = e_2 - e_1,\ \ r_3 = e_3 - e_2,$$
$$r_4 = e_4 - e_3,\ \ r_5 = e_5 - e_4,\ \ r_6 = e_6 - e_5,\ \ r_7 = e_7 - e_6,\ \ r_8 = e_8 - e_7\}$$

(a) Show that the roots of $E_7$ are those roots of $E_8$ that are orthogonal to $u = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 - e_8)$. Find the number of roots of the group $E_7$.
**Hint.** First show that the roots $\{r_i\}_{i=1}^7$ can be taken as the simple roots of $E_7$.

(b) Show that the roots of $E_6$ are those roots of $E_7$ that are orthogonal to $w = e_8 - e_7$. Find the number of roots of the group $E_6$.