Yale University Department of Mathematics
Math 235 Reflection Groups
Spring 2016

Problem Set # 8 (due at the beginning of class on Tuesday 19 April)

Reading: GB 5.1–5.3.

Problems:

1. Show that any linearly independent subset $v_1, \ldots, v_k$ of a Euclidean space V lies on one side of some hyperplane, i.e., that there exists $t \in V$ such that $\langle t, v_i \rangle > 0$ for all $1 \leq i \leq k$.
Hint. Consider t perpendicular to $v_i - v_1$ for all $2 \leq i \leq k$.

Use this to verify that for each Coxeter group $G \subset O(V)$ of type A_n, B_n, and D_n, there exists a vector $t \in V$ with respect to which the set of positive roots Δ^+ is as explained in class.

2. Let G be a crystallographic Coxeter group and Π be a base of the root system such that the $\mathbb{Z}$-linear span of Π is a G-invariant lattice. (We had given length conditions on the elements of Π to ensure this.) This endows the roots of the whole root system Δ with lengths.

(a) If G is an irreducible crystallographic Coxeter group, show that the orbits of G acting on Δ are the subsets consisting of roots of the same length.
Hint. If r_i and r_j are simple roots of the same length that are adjacent (in the Coxeter graph), show that $S_i S_j (r_i) = r_j$.

(b) What are the orbits of $G = H_2^n$ acting on its root system, when G is not crystallographic? In this case, assume that all roots have length 1 to do the calculation.
Hint. Consider the cases when n is odd and even separately).

3. Let $G \subset O(V)$ be an irreducible Coxeter group of orthogonal transformations in a Euclidean space V. Suppose that $R : V \to V$ is a linear transformation commuting with G, in the sense that $RT = TR$ for all $T \in G$.

(a) (Schur's lemma) Show that either $R = 0$ or R is invertible.
Hint. Show that the subspace $\ker R \subset V$ is G-invariant. Show that the orthogonal complement of a G-invariant subspace is also G-invariant. Show that the root of any reflection in G must lie in a G-invariant subspace.

(b) Show that each root of G is an eigenvector of R. In particular, show that R is diagonalizable.

(c) If λ is an eigenvalue of R, show that all eigenvalues of R are equal to λ, i.e., $R = \lambda I$ is a scalar matrix.
Hint: Show that R has the same eigenvalue for any two adjacent (in the Coxeter graph) simple roots of G, and use the fact that the Coxeter graph of an irreducible group is connected.

(d) Conclude that the center of G (the set of all elements $T \in G$ that commute with every element of G) is either trivial, or consists of just $\pm I$.

(e) What is the center of the irreducible Coxeter group of type B_n for $n \geq 2$?