Yale University Department of Mathematics

Math 235 Reflection Groups

Spring 2016

Problem Set # 7 (due at the beginning of class on Tuesday 12 April)

Reading: GB 5.1–5.2.

Problems:

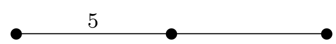
1. Let A be a real symmetric positive definite $n \times n$ matrix, and let P be a real $n \times k$ matrix, where $1 \le k \le n$. Suppose that P has rank k. Show that

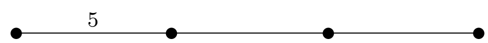
$$B = P^t A P$$

is a real symmetric positive definite $k \times k$ matrix.

2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with n distinct real eigenvalues. Suppose that $S : \mathbb{R}^n \to \mathbb{R}^n$ is another linear transformation that commutes with T, in the sense that $TS = ST$. Prove that S is diagonalizable.

3. The purpose of this exercise is to show that the Coxeter graphs

I_3 

I_4 

are positive definite.

(a) Show that the complex number $x = \cos(2\pi/5) + i \sin(2\pi/5)$ satisfies the equation

$$x^4 + x^3 + x^2 + x + 1 = 0$$

and hence

$$x^2 + x + 1 + 1/x + 1/x^2 = 0.$$

Hint. Draw x and its powers on the complex plane.

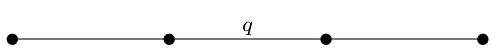
(b) Make the substitution $y = x + 1/x$ in the last equation and solve for y.

(c) Express $\cos(2\pi/5)$ and $\cos(\pi/5)$ in terms of y and obtain their numerical values in radicals.

(d) Show that the Coxeter graphs I_3 and I_4 are positive definite.
Hint. Use the following recursive formula to be proved in class: Letting G be a Coxter graph with vertex a_1 connected with only one other vertex a_2 by an edge marked p, and letting $G_1 = G \setminus \{a_1\}$ and $G_2 = G \setminus \{a_1, a_2\}$, then

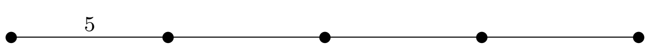
$$\det G = \det G_1 - \cos^2(\pi/p)\ \det G_2.$$

4. Use the recursive formula above to show that the determinant of the following graph is zero:

Z_4 

where $\cos(\pi/q) = 3/4$.

Also (using the values from 3c) show that the determinant of the following graph is negative:

Y_5' 

Deduce that if a graph contains a subgraph of type H_2^5, then it must be H_2^m with $m \ge 5$, I_3, or I_4.