Extra Credit Problem Set # 6 (due at the beginning of class on Tuesday 29 March)

Reading: GB 4.1–5.1.

Problems:

1. Consider the action of S_4 on the Euclidean space $\mathbb{R}^4$ by permutations of the standarnd orthonormal basis $\{e_1, e_2, e_3, e_4\}$. Let $V \subset \mathbb{R}^4$ be the subspace spanned by the vectors $r_i = e_i - e_{i+1}$ for $r = 1, 2, 3$. In the previous problem set, we found that S_4 acting on V is geometrically equivalent to a group of type $W]T$ and that it has a root system consisting of the 12 vectors

$$\Delta = \{\pm(e_i - e_j) \ : \ 1 \leq i < j \leq 4\}.$$

Here, we are not worrying about the lengths of roots.

(a) Let $v_0 = e_1 + e_2 + e_3 + e_4$ and $V_0 = \text{span}\{v_0\}$. Show that the $S_4 \subset O(\mathbb{R}^4)$ acts trivially on V_0 and that $V = V_0^{\perp}$. Does S_4 act effectively on V?

(b) Let $t = (2, 1, -1, -2) \in V$. Find Δ_t^+.

(c) Find the t-base Π_t and express each element of Δ_t^+ as a linear combination of the simple roots with nonnegative coefficients.

(d) Compute the angles between the roots in Π_t.

(e) Find the axes of the rotations of order 3 in $W]T$ (thought of as S_4 acting on V). **Hint.** What is the angle between two roots of reflections whose product is a rotation by $2\pi/3$?

(f) In fact, $W]T$ turns out to be the group of all orthogonal symmetries of a tetrahedron. In view of this, the axes found in the previous part are the lines connecting the vertices of the tetrahedron with its center. Use this observation to find the angles between the lines connecting the vertices with the center of a regular tetrahedron.

(g) Find the dual basis Π_t^*. Find the angles between the elements of the dual basis. Use Π_t^* to describe a fundamental region F for $W]T$.

(h) Write the Coxeter matrix of $W]T$ and draw its Coxeter graph.

2. Consider the Coxeter group $G = H_2^n$.

(a) If one of the roots is $(1, 0)$, find all roots of G.

(b) Let $t = (\sin \pi/4n, \cos \pi/4n)$. Find the positive roots Δ_t^+ and the simple roots Π_t.

(c) Find the dual basis $\Pi_t^* = \{s_1, s_2\}$. What is the angle between them? Use this to sketch a fundemantal region for H_2^n.

3. Find all reflections in the group $H_3^7]C_3^7$ and compute the subgroup G generated by the reflections. Determine if G is a Coxeter group, and if not, how G acts on $V_0(G)^{\perp}$. Identify the corresponding Coxeter group in the classification. Sketch its Coxeter graph.

Do the same for the group $C_3^{14}]C_3^7$.