Yale University Department of Mathematics
Math 235 Reflection Groups
Spring 2016

Problem Set # 5 (due at the beginning of class on Thursday 3 March)

Remark. Let G be a group. The subgroup generated by the empty set $\varnothing \subset G$ is the trivial subgroup $\{1\}$. Also, the trivial group is not a Coxeter group.

Reading: GB 4.1.

Problems:

1. *Who is generated by reflections?*

(a) Describe the subgroup generated by all reflections in (a group of type) $(C_3^3)^*$. Conclude that $(C_3^3)^*$ is not a Coxeter group. Is the subgroup generated by reflections in $(C_3^3)^*$ a Coxeter group?

(b) Describe the subgroup generated by all reflections in (a group of type) $(H_3^3)^*$. Conclude that $(H_3^3)^*$ is not a Coxeter group. Is the subgroup generated by reflections in $(H_3^3)^*$ a Coxeter group?

(c) Is $H_3^3]C_3^3$ generated by its reflections? Is it a Coxeter group? Why or why not?

2. *Conjugacy classes of permutations.*

(a) Use your results from the last problem set to show that two permutations in S_n are conjugate if and only if they have the same cycle type.

(b) Conclude that the number of conjugacy classes in S_n is equal to the number of partitions of n.

(c) How many conjugacy classes are there in S_6?

(d) Count the number of elements in each conjugacy class of S_6? (The sum of these numbers should be 6!.) Explain how you do the counting.

3. Consider the action of S_4 on the Euclidean space $\mathbb{R}^4$ by permutations of an orthonormal basis $\{e_1, e_2, e_3, e_4\}$.

(a) Show that this gives an injective map $S_4 \to O(\mathbb{R}^4)$, identifying a subgroup of $O(\mathbb{R}^4)$ isomorphic to S_4.

(b) Show that the vectors $\{r_i = e_i - e_{i+1}\}_{i=1}^3$ form a basis of a 3-dimensional subspace $V \subset \mathbb{R}^4$ that is invariant with respect to this action.

(c) Then S_4 acts on V. Verify that this gives an injective map $S_4 \to O(V)$, identifying a subgroup of $O(V)$ isomorphic to S_4.

(d) Identify this subgroup with one of the groups in the classification of the finite groups of orthogonal transformations in 3-dimensions.

(e) What is the root system of this group? How many roots does it contain? Give expressions for all roots as linear combinations of $\{e_1, e_2, e_3, e_4\}$.

(f) Find the angles between the roots.