Yale University Department of Mathematics
Math 235 Reflection Groups
Spring 2016

Problem Set # 4 (due at the beginning of class on Thursday 25 February)

Reading: GB 2.3–2.6.

Problems:

1. What does that star actually do?

(a) Show that the group $(C_3^n)^*$ is isomorphic to C_3^{2n} if and only if n is odd.

(b) Show that $(C_3^n)^*$ and C_3^{2n} are not geometrically equivalent for any $n \geq 1$.

2. Describe geometrically the action of each elements of the following groups:

(a) $(H_3^3)^*$

(b) $H_3^4]C_3^4$

3. For each of the following lattices in $\mathbb{R}^3$, check if they are invariant under the action of the groups $(H_3^3)^*$ and $H_3^4]C_3^4$.

(a) All integer linear combinations of the vectors

$$(1,0,0), \quad (-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0), \quad (0,0,1)$$

(b) All integer linear combinations of the vectors

$$(1,0,0), \quad (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0), \quad (0,0,3)$$

(c) All points with integer coordinates.

All coordinates are given in the standard orthonormal basis in $\mathbb{R}^3$.

4. Generators for the symmetric group.

(a) Prove the following generalization of a statement from class. If $\sigma \in S_n$ is any permutation and $\alpha = (a_1\, a_2\, \ldots\, a_k) \in S_n$ is a k-cycle, then $\sigma\alpha\sigma^{-1} = (\sigma(a_1)\, \sigma(a_2)\, \ldots\, \sigma(a_k))$.

(b) Prove that the n-cycle $(1\,2\,\ldots\,n)$ and the transposition $(1\,2)$ generate S_n.

5. Let G be a finite rotation group in $\mathbb{R}^3$ and $H \subset G$ a subgroup of index 2. Show that $G]H \cong G$ as abstract groups.

6. Let G be a group and $H \subset G$ a subgroup of index 2. Show that H is a normal subgroup of G.