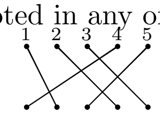
Yale University Department of Mathematics
Math 235 Reflection Groups
Spring 2016

Problem Set # 3 (due at the beginning of class on Thursday 18 February)

Permutations. A permutation of n elements is a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Under composition, they form a group, called the symmetric group S_n. There are many available notations to denote permutations. For example, the permutation σ of 5 elements such that $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 5$, $\sigma(4) = 1$, and $\sigma(5) = 3$ can be denoted in any of the following notations

$$\begin{pmatrix} 12345 \\ 24513 \end{pmatrix}$$



$$(124)(35)$$

The first is the "long-form" notation, the second is the "crossing pattern" notation, the third is the "cycle" notation. The first two are self-explanatory and quite natural, though cumbersome. The cycle notation is very powerful and compact, but might not seem intuitive at first. A **k-cycle** is a permutation of n elements that "cycles" through some subset of $\{1, \ldots, n\}$ of length k. For example, the permutation $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(4) = 1$, and fixing the rest $\sigma(3) = 3$ and $\sigma(5) = 5$, is a 3-cycle of 5 elements, which is denoted (124) in cycle notation. The simpliest cycles, the 2-cycles, are known as **transpositions**. In fact, every permutation can be expressable as a product of *disjoint* cycles.

Reading: GB 2.3–2.6.

Problems:

1. Let X_1 and X_2 be regular n-gons of the same size centered at the origin in $\mathbb{R}^2$. Show that there is an orthogonal transformation $T \in O(\mathbb{R}^2)$ such that $T(X_1) = X_2$. If H_1 and H_2 are the groups of rotations of X_1 and X_2, then show that $H_2 = TH_1T^{-1}$. If G_1 and G_2 are the groups of all orthogonal symmetries of X_1 and X_2, then show that $G_2 = TG_1T^{-1}$. Conclude that any two finite cyclic or dihedral subgroups of the same order in $O(\mathbb{R}^2)$ are conjugate. **Recall.** Two subgroups K_1 and K_2 of a group G are called **conjugate** if there exists $g \in G$ such that $K_1 = g^{-1}K_2g = \{g^{-1}kg \ : \ k \in K_2\}$.

2. Consider the action of the group of rotational symmetries T of a tetrahedron on the vertices of the tetrahedron. From this action, construct a homomorphism $T \to S_4$. Prove that this homomorphism is injective and that the image is a subgroup of index 2 in S_4. This subgroup is called the alternating subgroup $A_4 \subset S_4$. Write down all subgroups of A_4. Does A_4 have any normal subgroups?

3. Show that the group of rotational symmetries W of a cube acts on the set of diagonals of the cube. From this action, construct a homomorphism $W \to S_4$. Prove that this homomorphism is injective and defines an isomorphism $W \cong S_4$.

4. Recall that there are 15 axes for rotations of order 2 joining midpoints of opposite edges of an icosehedron. If l_1 is one such axis, then there are two others l_2 and l_3, that are perpendicular to l_1 and to each other. (You might have to stare at a physical icosahedron to see this!) The unordered triple (l_1, l_2, l_3) of such axes will be called an "orthogonal frame of axes" of the icosahedron.

(a) Show that there are 5 different orthogonal frames of axes of the icosahedron.

(b) Show that the axes appearing in an orthogonal frame of axes of the icosahedron are the axes of rotation for a group of type H_3^2, hence that H_3^2 occurs five times as a subgroup of the group of rotational symmetries I of an icosahedron.

(c) Show that I acts on the set of orthogonal frames of axes of the icosahedron. From this action, construct a homomorphism $I \to S_5$. Prove that this homomorphism is injective and that the image is a subgroup of index 2 in S_5. This subgroup is called the alternating sucgroup $A_5 \subset S_5$.