Yale University Department of Mathematics
Math 235 Reflection Groups
Spring 2016

Problem Set # 2 (due at the beginning of class on Thursday 11 February)

Permutations. A **permutation** of n elements is a bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$. The set of all permutations of n elements, together with the operation of composition, forms a group called the **symmetic group** S_n. Note that $|S_n| = n!$.

Reading: GB 2.1–2.5.

Problems:

1. Consider the dihedral group H_2^3 generated by the reflections in $\mathbb{R}^2$ given by the matrices

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad T = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

(a) Show that this group is isomorphic to the symmetric group S_3. First write a multiplication table for S_3. Try to match up S and T with generators for S_3 so that the relations amongst these generators match up.
(b) Give a geometric interpretation of this isomorphism. **Hint.** Consider an equilateral triangle in $\mathbb{R}^2$ centered at the origin.

2. Find the smallest subgroup of $O(\mathbb{R}^2)$ that contains
(a) A rotation by $2\pi/5$ and a rotation by $\pi/2$.
(b) The same two rotations and a reflection with respect to the line $y = x$.

3. The symmetric group S_3 acts on the set $\mathcal{P}$ of all nonempty subsets of $\{1, 2, 3\}$, e.g., the permutation cycling $1 \mapsto 2 \mapsto 3 \mapsto 1$ takes the subset $\{1, 3\}$ to $\{1, 2\}$. Find all orbits of this action; for each element of $\mathcal{P}$, find the stabilizer subgroup.

4. Let X be a set with n elements and order the elements $X = \{x_1, \dots, x_n\}$. Given an action of a group G on X define a map $\rho : G \to S_n$ by the relation $g \cdot x_i = x_{\rho(g)(i)}$. Prove that ρ is a homomorphism, called the permutation representation of the action.

5. Consider the dihedral group H_2^4 acting as the group of symmetries of a square in $\mathbb{R}^2$ centered at the origin. Then each element of H_2^4 acts on the set of vertices by permutations. This defines a permutation representation

$$\rho : \; H_2^4 \to S_4.$$

Prove that ρ is injective and find the image of ρ in S_4.

6. Show that the set B of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \quad a, b \in \mathbb{R}, \;\; a \neq 0$$

is a group with respect to the matrix multiplication.
Consider the following subsets in B:

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}_{a \neq 0} \quad U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}_{b \in \mathbb{R}} \quad K = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}_{a > 0, b \geq 0}$$

Which of D, U, K are subgroups of B? Which of them are normal?