Yale University Department of Mathematics
Math 235 Reflection Groups
Spring 2016

Problem Set # 1 (due at the beginning of class on Thursday 4 February)

Proof by induction. To prove a statement that depends on a natural number n, the method of mathematical induction can be used:

- "Base case." Show that the statement holds for $n = 0$ (or $n = 1$, as appropriate).
- "Induction step." Assuming that the statement holds for a given natural number n, show that it also holds for $n + 1$.

Then it follows that the statement holds for all natural numbers n.

Reading: GB 1.1–1.2, 2.1–2.2.

Problems:

1. The Euler characteristic formula

$$V - E + F = 2,$$

relates the number of vertices (V), edges (E) and faces (F) of a convex polyhedron. A regular polyhedron is a convex polyhedron whose faces are equal regular polygons, with the same number meeting at each vertex. Using only the Euler characteristic (and no arguments with angles), show that the only regular polyhedra whose faces have more than 3 edges are the cube and the dodecahedron.

2. Give an example of a bilinear form on $V = \mathbb{R}^2$ that is:

(a) Nondegenerate and not symmetric.
(b) Symmetric and degenerate.
(c) Symmetric and nondegenerate but not positive definite.

3. Consider the reflections in $\mathbb{R}^2$ given by the matrices

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad T = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Check that S and T are reflections and find their reflection axes. What group do they generate? Find all group elements and write down a multiplication table between them.
Hint. Since $S^2 = T^2 = I$, the only nontrivial elements of the group generated by S and T are the products where S and T alternate. Find all distinct elements of this form.

4. Let S be the reflection in $\mathbb{R}^2$ through the x-axis, and T the reflection in $\mathbb{R}^2$ through the line passing through the origin at $33°$ counterclockwise from the x-axis. Describe the group of orthogonal transformations generated by $\{S, T\}$, and identify it as wither cyclic or dihedral of a given order. **Hint.** You might brush up on your gcd and lcm.

5. Prove by induction that for all natural numbers $n \geq 0$, and a given angle $0 \leq \theta < 2\pi$, the following matrix equality holds:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}.$$

Give a geometric interpretation of this identity.