Yale University Department of Mathematics
**Math 235 Reflection Groups**
Spring 2016
Final Exam Problems

**Directions:** The final exam will take place in WLH 113 at 2:00 - 5:30 pm on Sunday, May 8, 2016. You will have 3 hours to complete the exam, which will consist of 5 problems randomly selected from those listed below. Additionally, you will have 30 minutes to check your work. No electronic devices will be allowed. No notes will be allowed. You will need to write your thoughts/proofs in a coherent way to get full credit.

**Reading:** GB 1.1–5.4.

**Problems:**

♠**1** (a) Suppose $G$ is generated by two reflections $S_1$ and $S_2$ in $\mathbb{R}^2$, such that the angle between the axes of the reflections is $12.5°$. What group do they generate? If it is finite, identify this group according to the classification of the finite groups in $O(\mathbb{R}^2)$, and give its order.
(b) What is the smallest subgroup of $O(\mathbb{R}^2)$ that contains rotations by $12°$ and by $45°$?

♠**2** Suppose that $\dim V$ is odd. Let $G \subset O(V)$ be a group of orthogonal transformations in $V$, and $H \subset G$ is the subgroup of all rotations in $G$. Suppose that $H \neq G$.
(a) Show that

$$K = H \cup \{-T : T \in G \setminus H\}$$

is a group that consists of rotations. Where did you use that $\dim V$ is odd?
(b) Can the set $G \setminus H$ have an element of odd order?

♠**3** Describe geometrically the action of all elements in the groups $C_3^4$ and $H_3^4$. For each of the groups, describe a fundamental region.

♠**4** List all groups of orthogonal transformations in $\mathbb{R}^3$ of order 24. Which of them contain(s) (a) an element of order 6, (b) an element of order 12?

♠**5** Consider the action of $S_n$ on the Euclidean space $\mathbb{R}^n$ by permutations of an orthonormal basis $\{e_1, e_2, \ldots e_n\}$.
(a) Show that $S_n$ acts effectively on the subspace orthogonal to the vector

$$e_1 + e_2 + \ldots + e_n.$$

(b) Find the root system corresponding to this Coxeter group. How many roots does it contain?

♠**6** Describe the subgroup generated by all the reflections in $H_3^8]H_3^4$. Is $H_3^8]H_3^4$ a Coxeter group? If so, describe the root system of the group. Otherwise, describe the root system of the subgroup in $H_3^8]H_3^4$ generated by the reflections.

♠**7** Can the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

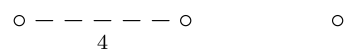
be an element of a finite group generated by reflections in $\mathbb{R}^4$, if it is
(a) written in a basis of simple roots of $G$,
(b) written in the standard orthonormal basis in $\mathbb{R}^4$?
Explain your answer.

♠**8** Suppose that a Coxeter group in $\mathbb{R}^3$ has a set of simple roots coinciding with an orthonormal basis $\{e_1, e_2, e_3\}$. What is the group? Find its order and describe its multiplicative structure. Identify this group according to the classification of finite groups of orthogonal transformations in $\mathbb{R}^3$.

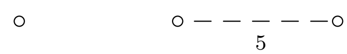
♢**1** Let $\{x_i\}_{i=1}^n$ be a basis in a Euclidean space $V$ and let $\{y_i\}_{i=1}^n$ be the dual basis, so that $\langle x_i, y_j\rangle = \delta_{ij}$. Suppose that $A$ is the matrix with $ij$th entry $\langle x_i, x_j\rangle$ and $B$ the matrix with $ij$th entry $\langle y_i, y_j\rangle$. Prove that $B = A^{-1}$. **Hint.** What is the decomposition of the vector $x_i$ in the basis $\{y_j\}_{j=1}^n$? What is the decomposition of the vector $y_i$ in the basis $\{x_j\}_{j=1}^n$?

♢**2** If $V$ is an $n$-dimensional Euclidean space and $\{x_1, \ldots x_{n+2}\}$ are $n+2$ vectors in $V$, show that $\langle x_i, x_j\rangle \geq 0$ for some $i \neq j$.

♢**3** Find a set of simple roots in $\mathbb{R}^3$ and identify the Coxeter groups corresponding to the Coxeter graphs




and




Identify each group according to the classification of all finite orthogonal groups in $\mathbb{R}^3$. Find the order of each group. Justify your answer.

♢**4** (a) Explain why the order of an irreducible simply laced (i.e., no markings greater than 3) Coxeter group of rank greater or equal to 3 is divisible by 24.
(b) Are the orders of all irreducible non-simply laced Coxeter groups of rank greater or equal to 3 divisible by 48? Explain your answer.

♢**5** List all Coxeter groups of rank 5 that have exactly 13 positive roots. Which of them are crystallographic?

♢**6** (a) For an irreducible Coxeter group of rank $n \geq 8$, what is the maximal number of mutually orthogonal simple roots?
(b) Of all irreducible Coxeter groups of rank $n = 8$, which have fewer than 8 mutually orthogonal roots?

♢**7** List all crystallographic Coxeter groups $G$ of rank 4 such that the action of $G$ on its root system has exactly 3 orbits. Justify your answer.

♢**8** Show that the Coxeter group $F_4$ contains as a subgroup (a) the group of symmetries of a regular tetrahedron, and (b) the group of symmetries of a cube.

♣**1** List all irreducible Coxeter groups of rank $\leq 4$ that have an element of order 5.

♣**2** Consider the Coxeter groups $B_n$, where $2 \leq n \leq 5$. Only one of these groups has the property that $B_n$ contains $A_n$ as a subgroup. Which is this group? Prove your answer.

♣**3** Let $G$ be a Coxeter group of rank $\geq 3$ and let $H \subset G$ be a subgroup generated by all but one simple reflections in $G$. Find all such pairs where $G$ is irreducible, and $H$ is commutative. In each case, identify $H$ as a Coxeter group.

♣**4** Consider the groups $T$, $W$, $W]T$, $W^*$, $T^*$.
(a) Which of these groups are isomorphic as abstract groups? Are they geometrically equivalent?
(b) List all pairs of groups where one is isomorphic to a subgroup of another as abstract groups.

♣**5** The candidate for the root system of $E_8$ is

$$\Gamma = \begin{cases} \pm e_i \pm e_j, & 1 \le i \ne j \le 8, \\ \frac{1}{2}\sum_{i=1}^{8}, \varepsilon_i e_i, & \varepsilon_i = \pm 1, \quad \prod_{i=1}^{8} \varepsilon_i = -1. \end{cases}$$

Show that the simple reflection $S_1$ corresponding to the simple root $r_1 = \frac{1}{2}(e_1 + e_2 + e_3 - e_4 - e_5 - e_6 - e_7 - e_8)$ preserves the set $\Gamma$.

♣**6** (a) Give an example of two roots $r_1$ and $r_2$ in the root system $\Delta$ of the irreducible Coxeter group $A_n$, $n \ge 2$, such that $r_1 + r_2$ is a root in $\Delta$, as well as an example of two roots $r_1'$ and $r_2'$, such that $r_1' + r_2'$ is not a root in $\Delta$.

(b) Formulate a sufficient condition for the sum of two roots of a Coxeter group to be a root, and prove it.

♣**7** (a) Give an example of two simple roots $r_i, r_j$ in a root system $\Delta$ of a crystallographic Coxeter group such that $r_i + r_j, r_i + 2r_j, r_i + 3r_j$ are roots in $\Delta$.

(b) For the root systems of type $A_n$, $n \ge 2$, what are the possible values of $k \in \mathbb{Z}$ such that $r_i + kr_j$ is a root for two simple roots $r_i$ and $r_j$?

♣**8** (a) Let $\Delta$ be a root system of an irreducible Coxeter group, and $\Pi$ a set of simple roots. Is there a root $r \in \Delta$ that is orthogonal to all simple roots in $\Pi$?

(b) Consider the root system of the Coxeter group $A_n$, $n \ge 2$. Can you find a root orthogonal to all but one simple roots of $A_n$?

♡**1** Consider the root system of the Coxeter group $B_n, n \ge 2$.

(a) Find a root $r$ of length 1 in this root system that is orthogonal to all but one simple root.

(b) Use Witt's theorem to find the stabilizer of $r$ in $B_n$.

(c) Use the formula

$$|B_n| = |\mathrm{Orb}(r)| \cdot |\mathrm{Stab}(r)|$$

to find the size of the orbit of $r$ under the action of $B_n$.

(d) Identify the vectors in the root system of $B_n$ that belong to the orbit of $r$.

♡**2** Consider the root system of the Coxeter group $B_n, n \ge 2$.

(a) Find a root $r'$ of length $\sqrt{2}$ in this root system that is orthogonal to all but one simple root.

(b) Use Witt's theorem to find the stabilizer of $r'$ in $B_n$.

(c) Use the formula

$$|B_n| = |\mathrm{Orb}(r')| \cdot |\mathrm{Stab}(r')|$$

to find the size of the orbit of $r'$ under the action of $B_n$.

(d) Identify the vectors in the root system of $B_n$ that belong to the orbit of $r$.

♡**3** Show the following inclusions of subgroups in Coxeter groups:

$$A_n \subset B_{n+1}, \quad D_n \subset B_n, \quad A_7 \subset E_8, \quad D_8 \subset E_8.$$

Which of these subgroups are normal? Justify your answer.

♡**4** Consider the Coxeter group generated by the reflections corresponding to every over node in the Coxeter graph of $A_n$.

(a) Identify this group as a Coxeter group. Is it normal in $A_n$?

(b) Let $n = 3$, consider the subgroup of $A_3 \equiv W]T$ generated by the reflections with respect to the first and the last node of the Coxeter graph. What is the order of this subgroup? Describe its action by symmetries of a regular tetrahedron.

♡**5** Recall that the center of an irreducible Coxeter group is either trivial, or consists of $\{\pm 1\}$. You may assume that for each of the groups $I_3, I_4, F_4$ it is possible to find a root $r \in \Delta$ such that $r$ is orthogonal to all but one simple root $r_i$, and that the stabilizer subgroup is $A_1 \times A_1$, $I_3$ and $B_3$, respectively.

(a) What is the center of the group $A_1 \times A_1$?

(b) What is the center of $B_3$?

(c) Use the fact that $-1 \in G$ if and only if $-1 \in \mathrm{Stab}(r)$ to find the centers of each of the groups $I_3, I_4, F_4$.

♡**6** Recall that the center of an irreducible Coxeter group is either trivial, or consists of $\{\pm 1\}$. You may assume that for each of the groups $E_6, E_7, E_8$ it is possible to find a root $r \in \Delta$ such that $r$ is orthogonal to all but one simple root $r_i$, and that the stabilizer subgroup in $E_6, E_7, E_8$ is $A_5$, $D_6$ and $E_7$, respectively.

(a) What are the centers of $A_5$ and $D_6$?

(b) Use the fact that $-1 \in G$ if and only if $-1 \in \mathrm{Stab}(r)$ to find the centers of each of the groups $E_6, E_7, E_8$.

♡**7** For an (irreducible) Coxeter group $G$ acting on a Euclidean space $V$, a Coxeter element $c \in G$ is a product of all simple reflections in some order. It turns out that Coxeter elements are all conjugate in $G$, and each one acts as rotation by $2\pi/|c|$ on a unique plane, called the Coxeter plane.

(a) Find all *distinct* Coxeter elements in the group $A_3$.

(b) Find $c^2$ for each Coxeter element and describe their action as symmetries of a regular tetrahedron.

(c) Describe the Coxeter planes of each $c$ in terms of the geometry of the regular tetrahedron.

♡**8** For an (irreducible) Coxeter group $G$ acting in a Euclidean space $V$, a Coxeter element $c \in G$ is a product of all simple reflections in any order. Show that the order of a Coxeter element in $A_n$ is $(n+1)$. **Hint.** Use the fact that $A_n \simeq S_{n+1}$ and write a Coxeter element as an element of the symmetric group.