Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2018

Problem Set # 5 (due in class Thursday 22 February)

Notation: Let V be an F-vector space. A linear transformation $T : V \to V$ is often called a **linear operator** on V. For $n > 0$, we write T^n for T composed with itself n times. For a matrix $A \in \mathsf{M}_{m\times n}(F)$, the **left multiplication transformation** is the linear map $L_A : F^n \to F^m$ defined by $L_A(v) = Av$, where we consider v as a column vector (or rather, as an $n \times 1$ matrix) and Av is the product of A and v.

Reading: FIS 2.2, 2.3

Problems:

1. FIS 2.2 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 2bce, 4, 5acdfg, 8, 9, 11 (Hint: Use §1.6 Corollary 2 part c), 13, 14.

2. FIS 2.3 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 4acd, 9, 11, 12 (For the second question in each part, if true, prove it, and if false then provide a counterexample), 16.

3. Let V be a vector space and $T : V \to V$ a linear operator.

(1) Prove that $T = T^2$ if and only if there exist subspaces W_0, W_1 of V and an internal direct sum decomposition $V = W_0 \oplus W_1$ such that T restricted to W_0 is the zero map and T restricted to W_1 is the identity map.

(2) Assume that V is finite dimensional. Prove that $T = T^2$ if and only if there exists an ordered basis β such that $[T]_\beta$ is a diagonal matrix whose diagonal entries are either 0 or 1.

Hint. FIS 2.3 exercises 16 and (the hint in) 17 will come in handy.

4. For $\theta \in \mathbb{R}$, consider the matrix

$$T_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in \mathsf{M}_{2\times 2}(\mathbb{R})$$

(1) For any θ, verify that T_θ is invertible and that $T_\theta^{-1} = T_{-\theta}$.

(2) Prove that $L_{T_\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ is counter-clockwise rotation by angle θ.
(Hint: Calculate how the slope of a nonzero vector changes.)

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