Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2018

Problem Set # 4 (due Thursday 15 February)

Notation: Let F be a field. For $A \in M_{n\times n}(F)$ a square matrix, define the **trace** $\mathrm{tr}(A) \in F$ to be the sum of the diagonal entries of A, i.e., if $A = (a_{ij})$ then $\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$. You can check that $\mathrm{tr} : M_{n\times n}(F) \to F$ is actually a linear map of F-vector spaces.

Let V be an F-vector space, $T : V \to V$ a linear map, and $W \subset V$ a subspace. Denote by $T|_W : W \to V$ the **restriction** of T to W, i.e., $T|_W(w) = T(w)$ for every $w \in W$. Then W is called **T-invariant** if $R(T|_W) \subset W$, i.e., T takes vectors from W to vectors in W. For example, $\{0\} \subset V$ and $V \subset V$ are always T-invariant subspaces for any linear map $T : V \to V$. As another example, the x-axis is T-invariant for the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, y)$.

Let W_1 and W_2 be F-vector spaces. Define the **cartesian product** $W_1 \times W_2$ to be the set of ordered pairs (w_1, w_2) of vectors $w_1 \in W_1$ and $w_2 \in W_2$. In fact, $W_1 \times W_2$ is an F-vector space under component-wise addition and scalar multiplication. For example, $F^2 = F \times F$.

Reading: FIS 2.1

Problems:

1. FIS 2.1 Exercises 1 (If true, then either cite or prove it, it false then provide a counterexample), 3, 5, 9, 11, 12, 14b, 16, 17, 18, 21, 25 (see definition before exercise 24), 26, 35, 36.

2. Let $W_1 \subset V$ and $W_2 \subset V$ be subspaces of an F-vector space V. Consider the map

$$\Sigma : W_1 \times W_2 \to W_1 + W_2$$

defined by $\Sigma(v, w) = v + w$. Verify that Σ is a linear map and is surjective. Prove that $W_1 + W_2 = W_1 \oplus W_2$ if and only if Σ is injective.

In that case, $\Sigma : W_1 \times W_2 \to W_1 \oplus W_2$ is bijective (i.e., injective and surjective), so you can think of the subspace $W_1 \oplus W_2$ "internal" to V as an object $W_1 \times W_2$ "external" to V.

3. Let $T : F^2 \to F^2$ be the linear transformation defined by $T(x, y) = (-y, x)$. For each choice of $F = \mathbb{F}_2$, $\mathbb{F}_p$ for p odd, $\mathbb{R}$, and $\mathbb{C}$, find all T-invariant subspaces of F^2.

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