Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2018

Problem Set # 10 (due in class Thursday, April 19)

Notation: You know about limits of real sequences $\{a_m\}$ from calculus. Basically, $\lim_{m\to\infty} a_m = L$ means that the terms a_m get arbitrarily close to L. For example, $\lim_{m\to\infty} \frac{1}{m} = 0$. Limits of sequences of matrices are simply taken component-wise. If $\{A_m\}$ is a sequence of $n \times p$ real matrices, then we say that $\lim_{m\to\infty} A_m = L$, where L is a real $n \times p$ matrix, if the ijth term of A_m has limit the ijth term of L, i.e., $\lim_{m\to\infty} (A_m)_{ij} = L_{ij}$. So, for example,

$$\lim_{m\to\infty} \begin{pmatrix} \frac{1}{m} & \frac{2m^2}{m^2+1} \\ 1+e^{-m} & \left(\frac{1}{2}\right)^m \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

Given a fixed matrix $A \in M_{n\times n}(\mathbb{R})$, we are often interested in the sequence of powers $\{A^m\}$ of A. FIS 5.3 tell you exactly when such a sequence is convergent, essentially all the eigenvalues must be 1 (and the eigenspace must have the right dimension) or have absolute value < 1.

There is also a method to calculate directly A^m, as long as A is diagonalizable. First diagonalize A, i.e., find a matrix $Q \in M_{n\times n}(\mathbb{R})$ so that $Q^{-1}AQ = D$ is a diagonal matrix. Then its easy to calculate D^m, it simply consists of raising each diagonal entry to the mth power. Then calculating powers $A^m = QD^mQ^{-1}$ only involves calculating an inverse and the product of three matrices.

Reading: FIS 5.3 (only pages 283–287), 6.1.

Problems:

1. FIS 5.3 Exercises 2bde, 4, 20, 21 (don't assume that e^D already exists, prove it!), 22.

Think about, but do not hand in: 23.

2. FIS 6.1 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 5, 6, 8, 10, 16, 17.

Think about, but do not hand in: 4, 11, 13, 19, 21.

3. Define the sequence $\{f_m\}$ of **Fibonacci numbers**

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

by the recursive formula $f_{m+2} = f_m + f_{m+1}$ for all $m \geq 0$. The goal of this problem is to derive the beautiful explicit formula

$$f_m = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right)$$

using sequences of matrices.

(a) Prove that for each $m \geq 0$, we have

$$\begin{pmatrix} f_{m+1} \\ f_m \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(b) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Show that the eigenvalues of A are $\frac{1\pm\sqrt{5}}{2}$. The larger of these eigenvalues is called the **golden ratio**.

(c) Diagonalize A, i.e., find a matrix $Q \in M_{2\times 2}(\mathbb{R})$ so that $Q^{-1}AQ = D$ is diagonal.

(d) For each $m \geq 0$, compute $A^m = QD^mQ^{-1}$.

(e) Derived the above explicit formula for the mth Fibonacci number f_m.

Yale University, Department of Mathematics, 10 Hillhouse Ave, New Haven, CT 06511
E-mail address: asher.auel@yale.edu