PRACTICE PROBLEM SOLUTIONS
MATH 225 SPRING 2018

1. Do Exercise 1 in all sections!

2. The characteristic polynomial of A is

$$\det(A-tI)=\det\begin{pmatrix}-\sqrt{3}/2-t & -1/2\\ 1/2 & -\sqrt{3}/2-t\end{pmatrix}=t^2+\sqrt{3}t+1$$

(Handy formula: $t^2-\mathrm{Tr}(A)t+\det(A)$.) Since the discriminant $(\sqrt{3})^2-4\cdot 1\cdot 1=-1$ of this quadratic polynomial is negative, it doesn't have real roots. In particular, there are no real eigenvalues.

However, considering A as a complex matrix (in particular, computing A^{25} doesn't depend on whether you think of A as real or complex), it does have eigenvalues. The quadratic formula, gives the roots of the characteristic polynomial as $\frac{-\sqrt{3}\pm i}{2}$. Since A is a 2×2 matrix with two distinct eigenvalues, A is diagonalizable. Let's solve for the eigenvectors:

$$A-\frac{-\sqrt{3}+i}{2}I=\begin{pmatrix}-i/2 & -1/2\\ 1/2 & -i/2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}i & 1\\ -1 & i\end{pmatrix}.$$

This matrix has rank 1 (as it should), so the columns are scalar multiples of each other, and we can readily pick off a generator of the null space. For example, $v=\begin{pmatrix}i\\1\end{pmatrix}$ works, so this is also an eigenvector. Similarly, an eigenvector for the other eigenvalue $\frac{-\sqrt{3}-i}{2}$ is seen to be $w=\begin{pmatrix}i\\-1\end{pmatrix}$. So the basis $\gamma=\{v,w\}$ diagonalizes A. Letting

$$Q=[I]_\gamma^\varepsilon=\begin{pmatrix}i & i\\ 1 & -1\end{pmatrix}$$

we have

$$Q^{-1}AQ=\begin{pmatrix}\frac{-\sqrt{3}+i}{2} & 0\\ 0 & \frac{-\sqrt{3}-i}{2}\end{pmatrix}=D.$$

Finally, notice that

$$\frac{-\sqrt{3}+i}{2}=\frac{-\sqrt{3}}{2}+\frac{1}{2}i=\cos(5\pi/6)+\sin(5\pi/6)i=e^{5\pi i/6}$$

and similary $\frac{-\sqrt{3}+i}{2}=e^{7\pi i/6}$. In particular, $\left(\frac{-\sqrt{3}\pm i}{2}\right)^{12}=1$. Hence $D^{12}=I$ and so $D^{25}=D$. Therefore $A^{25}=(QDQ^{-1})^{25}=QD^{25}Q^{-1}=QDQ^{-1}=A$.

3. The characteristic polynomial of A is $t^2-2t=t(t-2)$, which has distinct roots, hence A is diagonalizable. The eigenvalues are 0 and 2. The null space of A and $A-2I$ are generated by $(1,-1)$ and $(1,1)$, respectively, giving a basis of eigenvectors.

To compute the characteristic polynomial of B, we expand $\det(B - tI)$ by cofactors along the middle column, yielding $(1-t)(t^2-t) = -t(t-1)^2$. Hence the eigenvalues are 0 (of multiplicity 1) and 1 (of multiplicity 2). This implies that B has nullity 1 (you should know why). A generator for the nullspace of B can be spotted $(1, -1, 1)$ by looking for a linear dependence in the columns. To compute the 1-eigenspace, we see that the matrix

$$B - I = \begin{pmatrix} 1 & 0 & -2 \\ -1 & 0 & 2 \\ 1 & 0 & -2 \end{pmatrix}$$

has rank 1, hence nullity 2 (the same as the multiplicity), with null space generated by $(0, 1, 0)$ and $(2, 0, 1)$. So B is actually diagonalizable and we've already found a basis of eigenvectors.

To compute the characteristic polynomial of C, we first expand $\det(C - tI)$ by cofactors along the third row, eventually yielding $-t(-t(1-t)^2) = t^2(t-1)^2$. Hence the eigenvalues are 0 and 1 (both of multiplicity 2). The null space can be computed by spotting linear relations amongst the columns: $(1, 0, 0, -1)$ and $(0, 1, 1, -1)$; so good so far. To compute the 1-eigenspace, we see that the matrix

$$C - I = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix}$$

has rank 3 (for example, the first three columns are linearly independent). Hence it has nullity 1. Hence the 1-eigenspace is 1-dimensional but the eigenvalue has multiplicity 2. So C is not diagonalizable.

4. We see that the inner product of the first and fourth columns is $1/\sqrt{6} \neq 0$, so the matrix cannot be orthogonal. (Remember, the columns of an orthogonal matrix are an orthonormal basis of $\mathbb{R}^n$ with the standard dot product.) Normal means it commutes with its adjoint, i.e., its transpose (since it's a real matrix). Now use the handy formulas: the ijth entry if BB^t is $\langle R_i, R_j \rangle$, where R_i is the ith row of B; also the ijth entry of B^tB is $\langle C_i, C_j \rangle$, where C_i is the ith column of B. By inspection $13/12 = \langle R_1, R_1 \rangle \neq \langle C_1, C_1 \rangle = 1$, so B is not normal.

5. We apply Gram–Schmidt. Let the above vectors be v_1, v_2, v_3, respectively, written as rows. Let $u_1 = v_1$. Then

$$u_2 = v_2 - \frac{\langle v_2, v_1 \rangle}{||v_1||^2} v_1 = (-1, 4, 4, -1) - 6/4(1, 1, 1, 1) = (-5/2, 5/2, 5/2, -5/2).$$

Noting that multiplying by constants doesn't affect orthogonality, we can replace u_2 with $(-1, 1, 1, -1)$. Now

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{||u_1||^2} u_1 - \frac{\langle v_3, u_2 \rangle}{||u_2||^2} u_2 = (4, -2, 2, 0) - \frac{4}{4}(1, 1, 1, 1) - \frac{-4}{4}(-1, 1, 1, -1) = (2, -2, 2, -2).$$

Normalizing these, we obtain an orthonormal basis

$$\{u_1', u_2', u_3'\} = \{(1/2, 1/2, 1/2, 1/2), (-1/2, 1/2, 1/2, -1/2), (1/2, -1/2, 1/2, -1/2)\}.$$

6. The matrix, let's call it A, is symmetric and real, so is self-adjoint with respect to the standard dot product on $\mathbb{R}^3$. By the Spectral Theorem, we know there is an orthonormal basis of its eigenvectors. Already, we can see that $w_1 = (1, 1, 1)$ is an eigenvector with eigenvalue -1. In general, computing the characteristic polynomial

$$(1-t)^3 - 1 - 1 - 3(1-t) = -t^3 + 3t^2 - 4 = -(t-2)^2(t+1)$$

we see that the eigenvalues are -1 (with multiplicity 1) and 2 (with multiplicity 2). Trick: the fact that we already spotted an (-1)-eigenvector meant that we already knew that the characteristic polynomial was divisible by $(t+1)$. Since -1 has multiplicity 1, and we've already found an eigenvector, it generates the whole eigenspace. To compute the 2-eigenspace, we see that

$$A - 2I = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

has rank 1, hence nullity 2 (as it should), with null space generated by $w_2 = (1, -1, 0)$ and $w_3 = (1, 0, -1)$. So we've found a basis of eigenvectors. Now we need to make these orthonormal! Since A is normal, the eigenspaces for different eigenvalues are orthogonal, and indeed, we can see that $\langle w_1, w_2 \rangle = \langle w_1, w_3 \rangle = 0$. However, inside the 2-eigenspace, $\langle w_2, w_3 \rangle = 1$, so we need to find an orthogonal basis. We do Gram–Schmidt on $\{w_2, w_3\}$, modifying w_3 to $v_3 = w_3 - \frac{\langle w_3, w_2 \rangle}{\|w_2\|^2} w_2 = w_3 - \frac{1}{2} w_2 = (\frac{1}{2}, \frac{1}{2}, -1)$. Now that $\{w_1, w_2, v_3\}$ is orthogonal, we normalize to get an orthonormal basis of eigenvectors $\{\frac{1}{\sqrt{3}}(1, 1, 1), \frac{1}{\sqrt{2}}(1, -1, 0), \frac{1}{\sqrt{6}}(1, 1, -2)\}$.

7. We eliminate the xy-term by orthogonally diagonalizing the matrix

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

since $4x^2 + 2xy + 4y^2 = X^t A X$ where $X = \begin{pmatrix} x \\ y \end{pmatrix}$. Remembering the handy formula, A has characteristic polynomial $t^2 - 8t + 15 = (t-3)(t-5)$, so its eigenvalues are 3 and 5. To find the eigenvectors, we see that

$$A - 3I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad A - 5I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

each have rank 1 (as expected), and have null spaces generated by $w_1 = (1, -1)$ and $w_2 = (1, 1)$, respectively. Now we find an orthonormal basis of eigenvectors. Since A is symmetric, it is normal, hence eigenspaces for different eigenvalues are orthogonal. Indeed, $\langle w_1, w_2 \rangle = 0$, so we just need to normalize. So $\{\frac{1}{\sqrt{2}}(1, -1), \frac{1}{\sqrt{2}}(1, 1)\}$ is an orthonormal basis of eigenvectors. Hence the change of basis matrix $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ to this new basis will orthogonally diagonalize A. Indeed, $Q^t A Q = D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, equivalently, $A = QDQ^t$. Letting $X' = \begin{pmatrix} x' \\ y' \end{pmatrix} = Q^t X = \frac{1}{\sqrt{2}} \begin{pmatrix} x - y \\ x + y \end{pmatrix}$, then we have

$$4x^2 + 2xy + 4y^2 = X^t A X = X^t (QDQ^t) X = (X^t Q) D (Q^t X) = X'^t D X' = 3x'^2 + 5y'^2$$

So after this change of basis, we have $3x'^2 + 5y'^2 = 1$, which is a standard form for an ellipse (i.e., $(x'/a)^2 + (y'/b)^2 = 1$ with $a = \frac{1}{\sqrt{3}}$, $b = \frac{1}{\sqrt{5}}$). We can even rewrite the equation

$$4x^2 + 2xy + 4y^2 = \left(\frac{x - y}{\sqrt{2/3}}\right)^2 + \left(\frac{x + y}{\sqrt{2/5}}\right)^2 = 1.$$

8. Do this one on your own!