Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2018

Final Exam Review Sheet

Directions: The final exam will take place on Sunday, May 6, at 7:00 pm in DL 220. The exam is 3 hours long, though will have an extra 30 minutes to check your work. No electronic devices will be allowed. No notes will be allowed. On most problems, you will have to show your work to get full credit. One section will consist of true/false problems, for which you do not need to show your work.

Topics covered:

Some of the exercises of the form "prove this" are simply statements that you should know and not necessarily representative of problems on the final exam.

- Vector spaces and subspaces. Proving that a given subset of a vector space is a subspace. FIS 1.2 Exercises 1, 12, 13, 17, 18, 19. FIS 1.3 Exercises 13–16, 21, 22.
- Linear combinations and span. Verify if a given vector is a linear combination of other vectors. Solving systems of linear equations. FIS 1.4 Exercises 1, 3–5.
- Linear dependence and independence. Verify that a set of vectors is linearly (in)dependent. FIS 1.5 Exercises 1, 2, 7, 13, 17.
- Basis and dimension of a vector space. Choose a basis out of a generating set. Extend a linearly independent set to a basis. Dimensions of subspaces. FIS 1.6 Exercises 1–9, 13, 16, 17, 18, 23, 26, 27.
- Linear transformations. Null space and range. Rank-Nullity Theorem. Injective, surjective, and bijective linear maps. Calculate the dimension of a subspace. Calculate the rank of nullity of a linear map. Find bases for the null space and range of a linear map. FIS 2.1 Exercises 1–6, 9–12, 17, 18, 19.
- Matrix representation of a linear map. FIS 2.2 Exercises 1–5.
- Matrix multiplication and composition of linear maps. Left multiplication maps. FIS 2.3 Exercises 1, 3, 4, 9, 11, 12.
- Invertibility of linear maps. Isomorphism. Classification of finite dimensional vector spaces. FIS 2.4 Exercises 1, 2, 3, 5, 7, 11, 14, 15, 16, 18, 19.
- Change of coordinate matrix. FIS 2.5 Exercises 1–6.
- Elementary row and column operations. Elementary matrices. FIS 3.1 Exercises 1, 2, 3, 5, 12.
- Matrix rank. Matrix inverse. FIS 3.2 Exercises 1–7, 11, 19, 20, 21.

- Systems of linear equations $Ax = b$. Homogeneous and inhomogeneous systems. Systems with a unique solution. Consistent and inconsistent systems. FIS 3.3 Exercises 1–5, 7, 8, 9.

- Gaussian elimination. Reduced row echelon form and interpretation. Finding a basis in a generating set. Extending a linearly independent set to a basis. Computing the rank of a matrix. FIS 3.4 Exercises 1–14.

- Determinants. Determinants detect linear independence. FIS 4.1 Exercises 2, 3, 7, 8, 10. FIS 4.2 Exercises 1–22, 23, 25, 26, 27, 28, 29, 30. FIS 4.3 Exercises 9, 10, 11, 12, 13, 14, 15, 17, 19, 20, 21, 23, 24. FIS 4.4 Exercises 1–4, 5, 6.

- Eigenvalues and eigenvectors. Characteristic polynomial. Multiplicity of eigenvalues. Eigenspaces. Diagonalization and diagonalizability criteria. Powers of diagonalizable matrices. FIS 5.1 Exercises 1–4, 6–17, 20, 22, 23. FIS 5.2 Exercises 1–4, 7, 8, 11, 12, 13.

- Inner product spaces over the real and complex numbers. Standard inner products on $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathsf{M}_{n\times n}(\mathbb{R})$ and $\mathsf{M}_{n\times n}(\mathbb{C})$. Orthogonal vectors. Orthonormal basis. Gram–Schmidt orthogonalization process. FIS 6.1 Exercises 1–5, 9, 10, 16, 17, 19, 21, 23. FIS 6.2 Exercises 1–4, 5, 7, 8, 9, 13, 15, 17, 19.

- Adjoint of a linear transformation. Normal operators. Self-adjoint operators. Spectral theorem. Orthogonal operators. FIS 6.3 Exercises 1, 2, 3, 6–11, 13, 18. FIS 6.4 Exercises 1–4, 6, 9–12. FIS 6.5 Exercises 2abcdefg, 2, 5, 10, 11.

Assorted practice exam:

The following problems do not constitute an entire "practice final." They are meant to represent the material since the midterm.

1. There will be a true/false problem that will contain a selection from the "Exercise 1" of the above chapters.

2. Let A denote the real matrix

$$A = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$$

Find the characteristic polynomial of A and use it to either find all the eigenvalues of A or show none exist. Compute A^{25}.

3. Determine whether each of the following matrices is diagonalizable. For any that are, find a basis of eigenvectors.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & -2 \\ -1 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

4. Determine whether the following matrix is orthogonal and/or normal with respect to the standard inner product on $\mathbb{R}^4$.

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

5. Find an orthonormal basis for the following subspace of $\mathbb{R}^4$.

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix} \right\}$$

6. Find an orthonormal basis of $\mathbb{R}^3$, with respect to the standard inner product, consisting of eigenvectors of the following matrix:

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

7. Determine whether the following equation represents an ellipse or hyperbola and find a change of variables that puts the equation into standard form:

$$4x^2 + 2xy + 4y^2 = 1.$$

Recall that the standard form for an ellipse or hyperbola is:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$

respectively.

8. Let A denote the real matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors for A. Find an orthogonal matrix Q and a diagonal matrix D so that $A = QDQ^{-1}$. For any integer n, write an explicit equation for A^n.

Yale University, Department of Mathematics, 10 Hillhouse Ave, New Haven, CT 06511
E-mail address: asher.auel@yale.edu