Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2017

Midterm Exam Review Sheet

Directions: The midterm exam will take place in class on Thursday March 9th. You will have the entire class period, 1 hour and 15 minutes, to complete the exam. No electronic devices will be allowed. No notes will be allowed. On all problems, you will have to show your work to get full credit.

Topics covered and practice problems:

- Vector spaces and subspaces. Proving that a given subset of a vector space is a subspace. FIS 1.2 Exercises 12, 13, 17, 18, 19; FIS 1.3 Exercises 13–16, 21, 22.
- Linear combinations and span. Verify if a given vector is a linear combination of other vectors. Solving systems of linear equations. FIS 1.4 Exercises 3–5.
- Linear dependence and independence. Verify that a set of vectors is linearly (in)dependent. FIS 1.5 Exercises 2, 7, 13, 17.
- Basis and dimension of a vector space. Choose a basis out of a generating set. Extend a linearly independent set to a basis. Dimensions of subspaces. FIS 1.6 Exercises 2–9, 13, 16, 17, 18, 23, 26, 27.
- Linear transformations. Null space and range. Rank-Nullity Theorem. One-to-one and onto linear maps. Calculating the dimension of subspaces. Find bases for the null space and range of linear maps. FIS 2.1 Exercises 2–6, 9–12, 17, 18, 19.
- Matrix representation of a linear map. FIS 2.2 Exercises 2–5.
- Matrix multiplication. Left multiplication maps. FIS 2.3 Exercises 3, 4, 9, 11, 12.
- Invertibility of linear transformations and matrices. Isomorphism. Classification of finite dimensional vector spaces. FIS 2.4 Exercises 2, 3, 5, 7, 11, 14, 15, 16, 18, 19.
- Change of coordinate matrix. FIS 2.5 Exercises 2, 3, 4, 5, 6.

Practice exam questions:

1. Let $V = \mathsf{P}_3(\mathbb{R})$ be the $\mathbb{R}$-vector space of polynomials of degree at most 3 with real coefficients. Determine which of the following subsets of V are subspaces. For each, if it's a subspace, briefly explain why and calculate its dimension; if it's not, explain why not.

(1) $S_1 = \{p(x) \in V \ : \ p(0) = 0\}$
(2) $S_2 = \{p(x) \in V \ : \ p(1) = 0\}$
(3) $S_3 = \{p(x) \in V \ : \ p(0) = 1\}$
(4) $S_4 = \{p(x) \in V \ : \ p(0) = 0 \text{ and } p(1) = 0\}$
(5) $S_5 = \{p(x) \in V \ : \ p(0) = 0 \text{ or } p(1) = 0\}$
(6) $S_6 = \{p(x) \in V \ : \ p(0) + p(1) = 0\}$
(7) $S_7 = \{p(x) \in V \ : \ p(0)^2 + p(1)^2 = 0\}$

2. Bases of $\mathbb{R}^3$. There are many possible answers!

(1) Let $S_1 = \{(0,1,3),(1,2,3),(2,3,1),(1,1,2)\} \subset \mathbb{R}^3$. Find a subset of S that is a basis of $\mathbb{R}^3$. Express $(3,3,3)$ with respect to this basis.

(2) Let $S_2 = \{(1,1,1),(1,1,2)\} \subset \mathbb{R}^3$. Extend S_2 to a basis of $\mathbb{R}^3$. Express $(3,3,5)$ with respect to this basis.

3. Define a linear map $T : \mathsf{P}_2(\mathbb{R}) \to \mathsf{P}_3(\mathbb{R})$ by $T(p(x)) = (x-1)p(x)$. Find bases for the null space and range of T. Is T injective, surjective, and/or an isomorphism?

4. Let $C^\infty(\mathbb{R})$ be the vector space of differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Let $V \subset C^\infty(\mathbb{R})$ be the subspace generated by the functions e^x and xe^x.

(1) Prove that e^x and xe^x are linearly independent.

(2) The derivative defines a linear map $\frac{d}{dx} : V \to V$. What is its rank and nullity? Determine whether it's injective, surjective, and/or an isomorphism.

(3) With respect to the ordered basis $\beta = \{e^x, xe^x\}$ of V write the matrix representation $\left[\frac{d}{dx}\right]_\beta$ of the derivative map on V.

5. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Calculate $Q^{-1}AQ$. (Hint: You don't need to do any matrix multiplications!)

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