Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2017

Problem Set # 1 (due in class Thursday 26 January)

Notation: If S is a set of elements (numbers, vectors, rabbits, ...) then the notation "$s \in S$" means "s is an element of the set S." If T is another set, then the notation "$T \subseteq S$" means "every element of T is an element of S" or "T is a **subset** of S." For example, the set of squares is a subset of the set of rectangles.

We have notations for the following commonly referred to sets:

- $\mathbb{Z}$ is the set of integers (i.e., whole numbers, positive or negative).
- $\mathbb{Q}$ is the set of rational numbers (i.e., fractions $\frac{a}{b}$ for $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ with $b \neq 0$). It is a field, see Appendix C.
- $\mathbb{R}$ is the set of real numbers. It is a field, see Appendix C.
- $\mathbb{C}$ is the set of complex numbers (i.e., $a + bi$ for $a \in \mathbb{R}$ and $b \in \mathbb{R}$, where $i^2 = -1$). It is a field, see Appendix C.
- If F is any field then F^n is the set of ordered n-tuples $(a_1, a_2, \dots, a_n)$ where each $a_i \in F$ for $i = 1, \dots, n$. It is a vector space over F, see FIS 1.2 Example 1.

If S and T are sets, then a **function** $f : S \to T$ **from** S **to** T is the a rule that associates to each element $s \in S$, an element $f(s) \in T$. For example, $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ for all $x \in \mathbb{R}$, is a function. Another example, if S is the set of people in the room, $f : S \to \mathbb{Z}$ assigning to each person $p \in S$, their height $f(p) \in \mathbb{Z}$ in inches rounded up to the nearest inch, is a function.

Let S be a set and F be a field. Define $\mathcal{F}(S, F)$ to be the set of all functions $f : S \to F$. Then $\mathcal{F}(S, F)$ is a vector space over F by FIS 1.2 Example 3. The set of polynomials $\mathsf{P}(F)$ with coefficients in F is a vector space over F by FIS 1.2 Example 4. In fact, $\mathsf{P}(F) \subset \mathcal{F}(F, F)$ is a subspace. For each $n \geq 0$, the set of polynomials $\mathsf{P}_n(F)$ of degree at most n and with coefficients in F is also a vector space over F, and a subspace of $\mathsf{P}(F)$.

Reading: FIS 1.1–1.3

Problems:

1. FIS 1.2 Exercises 1 (For each statement, if it's true, either cite a Definition, Lemma, Proposition, Theorem, or Corollary from the book, or give a proof; if it is false, provide a counterexample), 9 (Hint: To prove that any zero vector is unique, suppose that 0 and $0'$ are zero vectors and then show using the zero vector axioms that $0 = 0'$), 10 (Hint: You can assume standard properties of differentiable functions and the results from §1.3), 13, 17.

2. FIS 1.3 Exercises 1 (See remark for exercise 1 above), 8abcf, 10, 11, 12, 20.

3. Let V be the set of positive real numbers. Define an addition $+_V$ by $x +_V y = xy$ for $x, y \in V$ and scalar multiplication $\cdot_V$ by $c \cdot_V x = x^c$ for $x \in V$ and $c \in \mathbb{R}$. Prove that $(V, +_V, \cdot_V)$ is an $\mathbb{R}$-vector space.

4. Prove that the set $\mathbb{Q}(\sqrt{5})$, of real numbers of the form $a + b\sqrt{5}$ for $a \in \mathbb{Q}$ and $b \in \mathbb{Q}$, is a field. (Hint: The most important field axiom you must verify is that every nonzero element of $\mathbb{Q}(\sqrt{5})$ has a multiplicative inverse.)

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