

Problem Set # 9 (due in class Thursday, April 16th)

**Notation:** You know about limits of real sequences  $\{a_m\}$  from calculus. Basically,  $\lim_{m \rightarrow \infty} a_m = L$  means that the terms  $a_m$  get arbitrarily close to  $L$ . For example,  $\lim_{m \rightarrow \infty} \frac{1}{m} = 0$ . Limits of sequences of matrices are simply taken component-wise. If  $\{A_m\}$  is a sequence of  $n \times p$  real matrices, then we say that  $\lim_{m \rightarrow \infty} A_m = L$ , where  $L$  is a real  $n \times p$  matrix, if the  $ij$ th term of  $A_m$  has limit the  $ij$ th term of  $L$ , i.e.,  $\lim_{m \rightarrow \infty} (A_m)_{ij} = L_{ij}$ . So, for example,

$$\lim_{m \rightarrow \infty} \begin{pmatrix} \frac{1}{m} & \frac{2m^2}{m^2+1} \\ 1 + e^{-m} & \left(\frac{1}{2}\right)^m \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

Given a fixed matrix  $A \in M_{n \times n}(\mathbb{R})$ , we are often interested in the sequence of powers  $\{A^m\}$  of  $A$ . FIS 5.3 tell you exactly when such a sequence is convergent, essentially all the eigenvalues must be 1 (and the eigenspace must have the right dimension) or have absolute value  $< 1$ .

There is also a method to calculate directly  $A^m$ , as long as  $A$  is diagonalizable. First diagonalize  $A$ , i.e., find a matrix  $Q \in M_{n \times n}(\mathbb{R})$  so that  $Q^{-1}AQ = D$  is a diagonal matrix. Then its easy to calculate  $D^m$ , it simply consists of raising each diagonal entry to the  $m$ th power. Then calculating powers  $A^m = QD^mQ^{-1}$  only involves calculating an inverse and the product of three matrices.

**Reading:** FIS 5.3 (only pages 283–287), 6.1.

### Problems:

1. FIS 5.3 Exercises 2bde, 4, 20, 21 (don't assume that  $e^D$  already exists, prove it!), 22.

Think about, but do not hand in: 23.

2. FIS 6.1 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 5, 6, 8, 10, 16, 17, 23abc (Here  $F^n$  has the standard inner product! Hint for part (c): prove that  $\langle Qx, Qy \rangle = \langle x, y \rangle$  first, then use the previous parts. To do this, expand in the standard basis.)

Think about, but do not hand in: 4, 11, 13, 19, 21.

3. Define the sequence  $\{f_m\}$  of **Fibonacci numbers**

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

by the recursive formula  $f_{m+2} = f_m + f_{m+1}$  for all  $m \geq 0$ . The goal of this problem is to derive the beautiful explicit formula

$$f_m = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^m - \left( \frac{1 - \sqrt{5}}{2} \right)^m \right)$$

using sequences of matrices.

(a) Prove that for each  $m \geq 0$ , we have

$$\begin{pmatrix} f_{m+1} \\ f_m \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(b) Let  $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ . Show that the eigenvalues of  $A$  are  $\frac{1 \pm \sqrt{5}}{2}$ . The larger of these eigenvalues is called the **golden ratio**.

(c) Diagonalize  $A$ , i.e., find a matrix  $Q \in M_{2 \times 2}(\mathbb{R})$  so that  $Q^{-1}AQ = D$  is diagonal.

(d) For each  $m \geq 0$ , compute  $A^m = QD^mQ^{-1}$ .

(e) Derived the above explicit formula for the  $m$ th Fibonacci number  $f_m$ .