Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2015

Problem Set # 8 (due in class Thursday April 9th)

Notation: Let $g(t)$ be a polyomial with coefficients in a field F. Let $T : V \to V$ be a linear operator on an F-vector space V. Then we define a linear operator $g(T) : V \to V$ as follows. First, for an integer $n \geq 1$, denote by $F^n : V \to V$ the n-fold composition

$$T^n = \underbrace{T \circ T \circ \cdots \circ T}_{n \text{ times}}$$

of T with itself. We define $T^0 = I_V$. Writing

$$g(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0,$$

then "g applied to the linear operator T" is defined to be

$$g(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0 I_V,$$

where the scalar multiples and sums are being taking inside the F-vector space $\mathcal{L}(V)$ of linear operators on V.

For an $n \times n$ matrix A with coefficients in F, there is an analogous definition of $g(A)$. Make sure you understand the content of the equality of linear operators $g(L_A) = L_{g(A)}$ on F^n.

Let $f(t)$ be the characteristic polynomial of a matrix $A \in \mathsf{M}_{n\times n}(F)$. Assume that $f(t)$ splits (i.e., is a product of linear factors) as

$$f(t) = (t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \cdots (t - \lambda_r)^{m_r}$$

where $\lambda_1, \dots, \lambda_r \in F$ are the eigenvalues of A. We call m_i the **multiplicity** of the eigenvalue λ_i.

Reading: FIS 5.1–5.2

Problems:

1. FIS 5.1 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 2bd, 3bd, 4e, 8, 9, 10, 11, 14, 17, 22, 23.

Think about, but do not hand in: 15, 16, 19, 20.

2. FIS 5.2 Exercises 9, 10, 11 (Hint: The easiest way to do part (a) is to do FIS 5.1 Exercise 16, which in turn means doing FIS 2.3 Exercise 13).

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