Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2015

Problem Set # 5 (due in class Thursday 26 February)

Notation: Let V be an F-vector space. A linear transformation $T : V \to V$ is often called a **linear operator** on V.

Note that if S and T are two finite sets and $f : S \to T$ is a bijection (i.e., f is injective and surjective), then S and T have the same number of elements.

Reading: FIS 2.4, 2.5

Problems:

1. FIS 2.4 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 9 (By "arbitrary matrices" they mean matrices that are not necessarily square), 10 (The definition of a matrix A being invertible is that there exists B such that both AB and BA are the identity, in this problem you prove that you only need to know one of these), 15, 17 (For the first part, use the restriction of T to V_0; for the second part, use exercise 15), 20.

2. FIS 2.5 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 2d, 3b, 6bd.

3. For $\theta \in \mathbb{R}$, consider the matrix

$$T_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$$

(1) For any θ, verify that T_θ is invertible and that $T_\theta^{-1} = T_{-\theta}$.

(2) Prove that $L_{T_\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ is counter-clockwise rotation by angle θ.
(Hint: Calculate how the slope of a nonzero vector changes.)

4. Let V be an $\mathbb{F}_p$-vector space of dimension n.

(1) Calculate the number of vectors in V. This is a function of p and n.
(Hint: You can you assume that $V = \mathbb{F}_p^n$. Why?)

(2) For each $1 \le k \le n$, calculate the number of k-tuples of linearly independent vectors in V. This is a function of p, n, and k.
(Hint: Start with a non-zero vector v_1, then choose v_2 not in the span of $\{v_1\}$, then choose v_3 not in the span of $\{v_1, v_2\}$, and so on, using the fact that you know the size of the span by the previous part.)

(3) Calculate the number of invertible $n \times n$ matrices over $\mathbb{F}_p$.
(Hint: Use the previous part.)

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