Yale University Department of Mathematics
**Math 225 Linear Algebra and Matrix Theory**
Spring 2015

Problem Set # 4 (due Thursday 19 February)

**Notation:** Let $F$ be a field. For $A \in M_{n\times n}(F)$ a square matrix, define the **trace** $\mathrm{tr}(A) \in F$ to be the sum of the diagonal entries of $A$, i.e., if $A = (a_{ij})$ then $\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$. You can check that $\mathrm{tr} : M_{n\times n}(F) \to F$ is actually a linear map of $F$-vector spaces.

Let $V$ be an $F$-vector space, $T : V \to V$ a linear map, and $W \subset V$ a subspace. Denote by $T|_W : W \to V$ the **restriction** of $T$ to $W$, i.e., $T|_W(w) = T(w)$ for every $w \in W$. Then $W$ is called **$T$-invariant** if $R(T|_W) \subset W$, i.e., $T$ takes vectors from $W$ to vectors in $W$. For example, $\{0\} \subset V$ and $V \subset V$ are always $T$-invariant subspaces for any linear map $T : V \to V$. As another example, the $x$-axis is $T$-invariant for the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, y)$.

Let $W_1$ and $W_2$ be $F$-vector spaces. Define the **cartesian product** $W_1 \times W_2$ to be the set of ordered pairs $(w_1, w_2)$ of vectors $w_1 \in W_1$ and $w_2 \in W_2$. In fact, $W_1 \times W_2$ is an $F$-vector space under component-wise addition and scalar multiplication. For example, $F^2 = F \times F$.

**Reading:** FIS 2.2, 2.3

**Problems:**

**1.** FIS 2.2 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 2bce, 4, 5acdfg, 8, 9, 11 (Hint: Use §1.6 Corollary 2 part c), 13.

**2.** FIS 2.3 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 4acd, 9, 11, 12 (If true, prove it, it false then provide a counterexample).

**3.** Let $W_1 \subset V$ and $W_2 \subset V$ be subspaces of an $F$-vector space $V$. Consider the map

$$\Sigma : W_1 \times W_2 \to W_1 + W_2$$

defined by $\Sigma(v, w) = v+w$. Verify that $\Sigma$ is a linear map and is onto. Prove that $W_1+W_2 = W_1 \oplus W_2$ if and only if $\Sigma$ is injective. In that case, remark that $\Sigma : W_1 \times W_2 \to W_1 \oplus W_2$ is bijective (i.e., injective and surjective), so you can think of the subspace $W_1 \oplus W_2$ "internal" to $V$ as an object $W_1 \times W_2$ "external" to $V$.

**4.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(x, y) = (-y, x)$. Find all $T$-invariant subspaces of $\mathbb{R}^2$.

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