Yale University Department of Mathematics
Math 225 Linear Algebra and Matrix Theory
Spring 2015

Problem Set # 3 (due in class Thursday 12 February)

Notation: (See page 22.) Let V be an F-vector space and let S_1 and S_2 be two nonempty subsets. Then the **sum** of S_1 and S_2, denoted $S_1 + S_2$, is the subset $\{x + y \;:\; x \in S_1 \text{ and } y \in S_1\} \subset V$.

We say that V is a **direct sum** (or an **internal direct sum** in some texts) of W_1 and W_2 if:

- both W_1 and W_2 are subspaces of V
- $W_1 \cap W_2 = \{0\}$
- $W_1 + W_2 = V$

and in this case we write $V = W_1 \oplus W_2$.

Reading: FIS 2.1 and also FIS 1.3 pages 22–23.

Problems:

1. FIS 2.1 Exercises 1 (If true, then either cite or prove it, it false then provide a counterexample), 3, 5, 9, 11, 12, 16, 17, 18, 21.

2. FIS 1.3 Exercises 23, 24, 28 (Hint: Do the 2×2 and 3×3 cases, do you see a pattern?), 30.

3. Let F be a field and $V = F^3$. Let $W \subseteq V$ be the subspace of vectors with zero component sum, i.e., vectors (a, b, c) such that $a + b + c = 0$. Let $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\} \subseteq V$.

(1) Prove that if the characteristic of F is not 2, then S is a basis for V.
(2) Prove that if the characteristic of F is 2, then S generates W. Find a subset of S that is a basis for W.

4. Let F be a field and V be the F-vector space of all infinite sequences $\{a_n\}$ of elements of F (see §1.2, Example 5). For each integer $i \geq 1$, let e_i be the sequence with 1 in the ith place and 0 in all other places. Prove that the set $S = \{e_i\}$ is linearly independent but is not a basis.

We will not cover this, but the results in §1.7 show that *any* vector space has a basis. So the set S in this problem is not a basis, but you might wonder what a basis of V might look like!

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