PRACTICE FINAL SOLUTIONS
MATH 225 SPRING 2015

1. Do Exercise 1 in all sections!

2. The characteristic polynomial of A is

$$\det(A - tI) = \det \begin{pmatrix} -\sqrt{3}/2 - t & -1/2 \\ 1/2 & -\sqrt{3}/2 - t \end{pmatrix} = t^2 + \sqrt{3}t + 1$$

(Handy formula: $t^2 - \mathrm{Tr}(A)t + \det(A)$.) Since the discriminant $(\sqrt{3})^2 - 4 \cdot 1 \cdot 1 = -1$ is negative, this polynomial doesn't have real roots. In particular, there are no real eigenvalues.

However, considering A as a complex matrix (in particular, computing A^{25} doesn't depend on whether you think of A as real or complex), it does have eigenvalues. The quadratic formula, gives the roots of the characteristic polynomial as $\frac{-\sqrt{3} \pm i}{2}$. Since A is a 2×2 matrix with two distinct eigenvalues, A is diagonalizable. Let's solve for the eigenvectors:

$$A - \frac{-\sqrt{3} + i}{2} I = \begin{pmatrix} -i/2 & -1/2 \\ 1/2 & -i/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}.$$

This matrix has rank 1 (as it should), so the columns are scalar multiples of each other, and we can readily pick off a generator of the null space. For example, $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ works, so this is also an eigenvector. Similarly, an eigenvector for the other eigenvalue $\frac{-\sqrt{3}-i}{2}$ is seen to be $w = \begin{pmatrix} i \\ -1 \end{pmatrix}$. So the basis $\gamma = \{v, w\}$ diagonalizes A. Letting

$$Q = [I]_\gamma^\varepsilon = \begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}$$

we have

$$Q^{-1}AQ = \begin{pmatrix} \frac{-\sqrt{3}+i}{2} & 0 \\ 0 & \frac{-\sqrt{3}-i}{2} \end{pmatrix} = D.$$

Finally, notice that

$$\frac{-\sqrt{3} + i}{2} = \frac{-\sqrt{3}}{2} + \frac{1}{2} i = \cos(5\pi/6) + \sin(5\pi/6)i = e^{5\pi i/6}$$

and similary $\frac{-\sqrt{3}+i}{2} = e^{7\pi i/6}$. In particular, $\left(\frac{-\sqrt{3}\pm i}{2}\right)^{12} = 1$. Hence $D^{12} = I$ and so $D^{25} = D$. Therefore $A^{25} = (QDQ^{-1})^{25} = QD^{25}Q^{-1} = QDQ^{-1} = A$.

3. The characteristic polynomial of A is $t^2 - 2t = t(t-2)$, which has distinct roots, hence A is diagonalizable. The eigenvalues are 0 and 2. The null space of A and $A - 2I$ are generated by $(1, -1)$ and $(1, 1)$, respectively, giving a basis of eigenvectors.

Date: May 4, 2015.

To compute the characteristic polynomial of B, we expand the $\det(B-tI)$ by cofactors along the middle column, yielding $(1-t)(t^2-t) = -t(t-1)^2$. Hence the eigenvalues are 0 (of multiplicity 1) and 1 (of multiplicity 2). This implies that B has nullity 1 (think about it!). A generator for the nullspace of B can be spotted $(1,-1,1)$ by looking for a linear dependence in the columns. To compute the 1-eigenspace, we see that the matrix

$$B - I = \begin{pmatrix} 1 & 0 & -2 \\ -1 & 0 & 2 \\ 1 & 0 & -2 \end{pmatrix}$$

has nullity 2 (the same as the multiplicity), with null space generated by $(0,1,0)$ and $(2,0,1)$. So B is actually diagonalizable and we've already found a basis of eigenvectors.

To compute the characteristic polynomial of C, we first expand the $\det(C-tI)$ by cofactors along the third row, eventually yielding $-t(-t(1-t)^2) = t^2(t-1)^2$. Hence the eigenvalues are 0 and 1 (both of multiplicity 2). The null space can be computed by spotting linear relations amongst the columns: $(1,0,0,-1)$ and $(0,1,1,-1)$. To compute the 1-eigenspace, we see that the matrix

$$C - I = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix}$$

has rank 3 (for example, the first three columns are linearly independent). Hence it has nullity 1. Hence the 1-eigenspace is 1-dimensional but the eigenvalue has multiplicity 2. So C is not diagonalizable.

4. We see that the inner product of the first and fourth columns is $1/\sqrt{6} \neq 0$, so the matrix cannot be orthogonal. (Remember, the columns of an orthogonal matrix are an orthonormal basis of $\mathbb{R}^n$ with the standard dot product.) Normal means it commutes with its adjoint, i.e., its transpose (since it's a real matrix). Now use the handy formulas: the ijth entry if BB^t is $\langle R_i, R_j \rangle$, where R_i is the ith row of B; also the ijth entry of B^tB is $\langle C_i, C_j \rangle$, where C_i is the ith column of B. By inspection $13/12 = \langle R_1, R_1 \rangle \neq \langle C_1, C_1 \rangle = 1$, so B is not normal.

5. We apply Gram-Schmidt. Let the above vectors be v_1, v_2, v_3 respectively. Let $u_1 = v_1$. Then $u_2 = v_2 - \frac{\langle v_2, v_1 \rangle}{||v_1||^2} v_1 = (-1,4,4,-1) - 6/4(1,1,1,1) = (-5/2, 5/2, 5/2, -5/2)$. Noting that multiplying by constants doesn't affect orthogonality, we can replace u_2 with $(-1,1,1,-1)$. Now $u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{||u_1||^2} u_1 - \frac{\langle v_3, u_2 \rangle}{||u_2||^2} u_2 = (4,-2,2,0) - \frac{4}{4}(1,1,1,1) - \frac{-4}{4}(-1,1,1,-1) = (2,-2,2,-2)$. Normalizing these, we obtain an orthonormal basis $\{u_1', u_2', u_3'\} = \{(1/2,1/2,1/2,1/2), (-1/2,1/2,1/2,-1/2), (1/2,-1/2,1/2,-1/2)\}$.

6. The matrix, let's call it A, is symmetric and real, so is self-adjoint with respect to the standard dot product on $\mathbb{R}^3$. By the Spectral Theorem, we know there is an orthonormal basis of its eigenvectors. Already, we can see that $w_1 = (1,1,1)$ is an eigenvector with eigenvalue -1. In general, computing the characteristic polynomial

$$(1-t)^3 - 1 - 1 - 3(1-t) = -t^3 + 3t^2 - 4 = -(t-2)^2(t+1)$$

we see that the eigenvalues are -1 (with multiplicity 1) and 2 (with multiplicity 2). Trick: the fact that we already spotted an (-1)-eigenvector meant that we already knew that the characteristic polynomial was divisible by $(t+1)$. Since -1 has multiplicity 1, and we've already found an eigenvector, it generates the whole eigenspace. To compute the 2-eigenspace, we see that

$$A - 2I = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

has nullity 2 (as it should), with null space generated by $w_2 = (1, -1, 0)$ and $w_3 = (1, 0, -1)$. So we've found a basis of eigenvectors. Now we need to make these orthonormal! Since A is normal, the eigenspaces for different eigenvalues are orthogonal, and indeed, we can see that $\langle w_1, w_2 \rangle = \langle w_1, w_3 \rangle = 0$. However, inside the 2-eigenspace, $\langle w_2, w_3 \rangle = 1$, so we need to find an orthogonal basis. We do Gram-Schmidt on $\{w_2, w_3\}$, modifying w_3 to $v_3 = w_3 - \frac{\langle w_3, w_2 \rangle}{\|w_2\|^2} w_2 = w_3 - \frac{1}{2} w_2 = (\frac{1}{2}, \frac{1}{2}, -1)$. Now that $\{w_1, w_2, v_3\}$ is orthogonal, we normalize to get an orthonormal basis of eigenvectors $\left\{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(1,-1,0), \frac{1}{2}\sqrt{\frac{3}{2}}(1,1,-2)\right\}$.

7. We eliminate the xy-term by orthogonally diagonalizing the matrix

$$A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

since $4x^2 + 2xy + 4y^2 = X^t A X$ where $X = \begin{pmatrix} x \\ y \end{pmatrix}$. Remembering the handy formula, A has characteristic polynomial $t^2 - 8t + 15 = (t-3)(t-5)$, so its eigenvalues are 3 and 5. To find the eigenvectors, we see that

$$A - 3I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad A - 5I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

each have rank 1 (as expected), and have null spaces generated by $w_1 = (1, -1)$ and $w_2 = (1, 1)$, respectively. Now we find an orthonormal basis of eigenvectors. Since A is symmetric, it is normal, hence eigenspaces for different eigenvalues are orthogonal. Indeed, $\langle w_1, w_2 \rangle = 0$, so we just need to normalize. So $\left\{\frac{1}{\sqrt{2}}(1,-1), \frac{1}{\sqrt{2}}(1,1)\right\}$ is an orthonormal basis of eigenvectors. Hence the change of basis matrix $Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ to this new basis will orthogonally diagonalize A. Indeed, $Q^t A Q = D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, equivalently, $A = QDQ^t$. Letting $X' = \begin{pmatrix} x' \\ y' \end{pmatrix} = Q^t X = \frac{1}{\sqrt{2}}\begin{pmatrix} x+y \\ -x+y \end{pmatrix}$, then we have

$$4x^2 + 2xy + 4y^2 = X^t A X = X^t (QDQ^t) X = (X^t Q) D (Q^t X) = X'^t D X' = 3x'^2 + 5y'^2$$

So after this change of basis, we have $3x'^2 + 5y'^2 = 1$, which is a standard form for an ellipse (i.e., $(x'/a)^2 + (y'/b)^2 = 1$ with $a = \frac{1}{\sqrt{3}}$, $b = \frac{1}{\sqrt{5}}$). We can even rewrite the equation

$$4x^2 + 2xy + 4y^2 = \left(\frac{x+y}{\sqrt{2/3}}\right)^2 + \left(\frac{-x+y}{\sqrt{2/5}}\right)^2 = 1.$$