PRACTICE MIDTERM #1

SOLUTIONS BY MAX EHRMAN

Exercise 1

(1), (2) S_1, S_2 are the null spaces of the maps $T : V \to \mathbb{R}$ given by $T(f) = f(a)$ where $a = 0, 1$ respectively. As the rank of this map is 1 (the constant polynomials map onto $\mathbb{R}$, for example), and the dimension of V is 4, by the Rank-Nullity Theorem, the dimension of S_1 and S_2 is three.

(3) S_3 is not a subspace as it does not contain 0, for example.

(4) S_4 is a subspace, as it is the intersection of S_1 and S_2. It is of dimension two: $\{x(x-1), x^2(x-1)\}$ is a basis.

(5) S_5 is not a subspace, since it is the union of two subspaces ($S_5 = S_1 \cup S_2$) and neither contains the other (to refer to an exercise many of you have seen in recitation). Directly, we have $x \in S_5$ and $x - 1 \in S_5$ but $f(x) = x + (x-1) = 2x - 1 \notin S_5$, as $f(0) = -1$ and $f(1) = 1$.

(6) S_6 is a subspace, as it is the null space of the linear map $T : V \to V$ given by $T(f) = f(0) + f(1)$ (you should check that this is a linear map).

(7) S_7 is a subspace, since the equation $p(0)^2 + p(1)^2 = 0$ is equivalent to $p(0) = 0$ and $p(1) = 0$ (so in fact, $S_7 = S_4$). Since $p(0)$ and $p(1)$ are real numbers, their squares are nonnegative, and hence the sum of their squares can only be zero when both are zero.

Exercise 2

(1) The subset $\{(0,1,3), (1,2,3), (2,3,1)\} = \{v_1, v_2, v_3\}$ is a basis. To prove this, first notice by inspection that v_1 and v_2 are not scalar multiples of each other, and are nonzero, so $\{v_1, v_2\}$ is a linearly independent set. Now, in order to show that we may add v_3 to this set without losing linear independence, it is sufficient to show that $v_3 \notin \text{span}(\{v_1, v_2\})$. Suppose by contradiction that we have $v_3 = av_1 + bv_2$, i.e. $v_3 \in \text{span}(\{v_1, v_2\})$. Then equating coordinates, we have $2 = b, 3 = a + 2b$, and $1 = 3a + 3b$. Substituting $b = 2$ in the second equation yields $3 = a + 4 \Rightarrow a = -1$, and plugging both of these into the third equation yields $1 = 3(-1) + 3(2) = 3$, a contradiction in the field $\mathbb{R}$. Therefore the set $\{v_1, v_2, v_3\}$ is linearly independent. As it has size three, and we know the dimension of $\mathbb{R}^3$ is 3, it is a basis.

To express $(3,3,3)$ in terms of this basis, we solve the equations $a(0,1,3)+b(1,2,3)+c(2,3,1)=(3,3,3)$, i.e. $b+2c=3, a+2b+3c=3, 3a+3b+c=3$. Subtracting the second from the first, we get $a+b+c=0$. Subtracting three times this from the third equation, we find $-2c=3$, so $c=-3/2$. The original first equation $b+2c=3$ now gives $b=6$, and the equation $a+b+c=0$ gives $a=-9/2$.

(2) To extend this linearly independent set to a basis, it suffices to add any vector outside of $\text{span}(S_2)$. Notice that the first two coordinates of each element in S_2 are equal. Thus any linear combination of elements in S_2 will have the first two coordinates equal. So, the vector $(1,0,0)$, for example, does not lie in $\text{span}(S_2)$, and so extends S_2 to a basis. By inspection, $(3,3,5)=(1,1,1)+2(1,1,2)$.

Exercise 3

$N(T)=\{f\in \mathsf{P}_2(\mathbb{R}) : (x-1)f=0\}=\{0\}$, as the degree of $(x-1)f(x)$ is at least one unless $f(x)=0$. Therefore T is one-to-one and has $\text{nullity}(T)=0$ and $\text{rank}(T)=3$. This implies that the images of vectors forming a basis for $\mathsf{P}_2(\mathbb{R})$ will be a basis for the range (they will generate, and there are the correct number of them). Therefore $\{T(1),T(x),T(x^2)\}=\{(x-1),(x-1)x,(x-1)x^2\}$ is a basis for the range of T.

Exercise 4

(1) If e^x and xe^x were linearly dependent, there would exist some scalar $c\in\mathbb{R}\backslash\{0\}$ such that $cxe^x=e^x$. As the right-hand side is always positive, and the left-hand side can be negative regardless of the value of c, this is impossible.

(2) Let $f(x)=ae^x+bxe^x$ be an element of V. Then $f(x)\in N\big(\frac{d}{dx}\big)$ if and only if

$$\frac{df}{dx}=ae^x+be^x+bxe^x=0=(a+b)e^x+bxe^x=0.$$

Since e^x and xe^x are linearly independent, we see that $f(x)\in N\big(\frac{d}{dx}\big)$ if and only if $a+b=0$ and $b=0$. As the only solutions are $a=b=0$, we have that $N\big(\frac{d}{dx}\big)=0$. Once again, $\frac{d}{dx}: V\to V$ is one-to-one, has nullity 0, rank 2, and a basis for the range is given by the images of the basis vectors of the domain, i.e. $\{e^x, e^x+xe^x\}$. But since $\text{span}(\{e^x,e^x+xe^x\})$=$\text{span}(\{e^x,xe^x\})=V$, we see that this map is also onto.

(3) As we have $\frac{d}{dx}e^x=e^x$ and $\frac{d}{dx}xe^x=e^x+xe^x$, the matrix is $\begin{pmatrix}1&1\\0&1\end{pmatrix}$.