MIDTERM 2 PRACTICE EXAM SOLUTIONS

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1. *Find a cubic polynomial $p(x)$ with real coefficients such that $p(-2) = 0$, $p(-1) = 4$, $p(1) = 0$, and $p(2) = 4$.*

Plugging in these values to $p(x) = ax^3 + bx^2 + cx + d$ have the following system of equations:

$$\begin{aligned} -8a + 4b - 2c + d &= 0 \\ -a + b - c + d &= 4 \\ a + b + c + d &= 0 \\ 8a + 4b + 2c + d &= 4 \end{aligned}$$

which we solve by reducing an augmented matrix:

$$\left(\begin{array}{cccc|c} -8 & 4 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 & 4 \\ 1 & 1 & 1 & 1 & 0 \\ 8 & 4 & 2 & 1 & 4 \end{array}\right) \to \left(\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 4 \\ 1 & 1 & 1 & 1 & 0 \\ -8 & 4 & -2 & 1 & 0 \\ 0 & 8 & 0 & 2 & 4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -1 & 1 & -1 & -4 \\ 0 & 2 & 0 & 2 & 4 \\ 0 & -4 & 6 & -7 & -32 \\ 0 & 8 & 0 & 2 & 4 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 6 & -3 & -24 \\ 0 & 0 & 0 & -6 & -12 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

so the polynomial is $p(x) = x^3 - 3x + 2$

2. *Consider the system of linear equations with real coefficients:*

$$\begin{aligned} x + 2y - z &= 1 \\ 2x + 3y + z &= 3 \\ x + 3y + az &= 0 \\ x + y + 2z &= b \end{aligned}$$

i) Find values of the parameters a and b for which the system has no solution.
ii) Find values of the parameters a and b for which the system has a unique solution, and find this solution.
iii) Find values of the parameters a and b for which the system has infinitely many solutions, and find all solutions of the resulting system.

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We row reduce the augmented matrix as follows, treating a and b as unknown scalars:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 1 & 3 \\ 1 & 3 & a & 0 \\ 1 & 1 & 2 & b \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & a+1 & -1 \\ 0 & -1 & 3 & b-1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & a+4 & 0 \\ 0 & 0 & 0 & b-2 \end{array}\right)$$

i) We thus see that in order to have any solutions (a consistent system) we are forced to take $b = 2$, and that our choice of a doesn't affect whether or not we have solutions. Therefore we have no solutions precisely when $b \neq 2$.

ii) In order to generate a unique solution, we need the matrix of coefficients to have rank 3, which is seen in the last augmented matrix to be equivalent to $a + 4 \neq 0$, i.e., $a \neq -4$ (we still need to demand $b = 2$ to have any solutions at all). In this case the last equation gives $z = 0$, giving $x = 3$ and $y = -1$.

iii) If $b = 2$ and $a = -4$, the resulting system only has rank 2 and so has an extra degree of freedom, resulting in infinitely many solutions. In this case we are reduced to only $x + 5z = 3$ and $y - 3z = -1$. We can see a particular solution $(x, y, z) = (3, -1, 0)$. The solutions to the associated homogeneous system $x + 5z = 0$ and $y - 3z = 0$ are given by $\{(-5z, -3z, z) : z \in \mathbb{R}\} = \operatorname{span}\big((-5, -3, 1)\big)$. Hence the solution set to the original inhomogeneous system is $\{(3 - 5z, -1 + 3z, z) : z \in \mathbb{R}\}$.

3. *Let*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Calculate $Q^{-1}AQ$. (Hint: You don't need to do any matrix multiplications!)

Let $\varepsilon = \{e_1, e_2, e_3, e_4, e_5\}$ be the standard ordered basis of $\mathbb{R}^5$ and let $\gamma = \{\gamma_1, \cdots, \gamma_5\} = \{e_2, e_4, e_5, e_1, e_3\}$ be a different ordered basis, just given by a permutation of the basis vectors. Then Q is the change of basis matrix $[I_5]_\gamma^\varepsilon$, and so $Q^{-1}AQ$ will simply be $[A]_\gamma$ (see Theorem 2.23 and its Corollary).

So we need to compute the matrix representation $[A]_\gamma$. For simplicity of notation, let $v = e_1 + e_2 + e_3 + e_4 + e_5$ (represented by the vector of all 1's). Then $A(\gamma_1) = A(e_2) = v - e_3 = v - \gamma_5$, so the first column of $[A]_\gamma$ is $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. Similarly we compute $[A]_\gamma = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$.

4. *Calculate the ranks of the following real matrices:*

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 1 & -2 & 3 & 0 \\ 0 & 0 & 4 & 8 \\ 2 & 4 & 0 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 & 3 & 6 \\ 1 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 40 \\ 3 & 0 & 6 & 7 & 15 \end{pmatrix}$$

For A, we row reduce

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 1 & -2 & 3 & 0 \\ 0 & 0 & 4 & 8 \\ 2 & 4 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & -4 & 3 & -3 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

from which we can see that if we keep row reducing we will obtain pivots in the first three columns, so A has rank 3.

Another way, without row-reducing, is to notice that the 4th row is twice the first row, so the matrix has rank ≤ 3. If we can then find a 3×3 submatrix with nonzero determinant, A will have rank 3 by Exercise 23 in Section 4.3. For instance, the upper left most 3×3 submatrix has determinant -16.

As for B, we can partially row reduce:

$$B = \begin{pmatrix} 1 & 0 & 2 & 3 & 6 \\ 1 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 40 \\ 3 & 0 & 6 & 7 & 15 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & 3 & 6 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 34 \\ 0 & 0 & 0 & -2 & -3 \end{pmatrix}$$

Rather than continuing to row reduce, we notice that the first two rows will result in pivots in the first and third columns, and the last two rows span the subspace generated by $\{e_4, e_5\}$. Therefore the rank of this matrix is 4. Already, seeing the column (or row) of zeros means that the rank must be ≤ 4.

5. *If possible, find a subset of the vectors* $(1, 4, -2)$*,* $(-3, -5, 8)$*,* $(1, 37, -17)$*,* $(-8, 12, -4)$*, and* $(2, -3, 1)$ *that are a basis for* $\mathbb{R}^3$*. If not, prove that it is impossible.*

Extend $(2, 1, 3, 4)$*,* $(1, 1, 1, 3)$ *to a basis of* $\mathbb{R}^4$*.*

We can use the procedure outlined on page 192–193 (Section 3.4, Example 3). We need to partially row reduce the matrix whose columns are the given vectors. Then hopefully, three pivots will emerge, from which we can extract three linearly independent columns of the original matrix. The row reduction starts out:

$$\begin{pmatrix} 1 & -3 & 1 & -8 & 2 \\ 4 & -5 & 37 & 12 & -3 \\ -2 & 8 & -17 & -4 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 1 & -8 & 2 \\ 0 & 7 & 33 & 44 & -11 \\ 0 & 0 & -171 & -228 & 57 \end{pmatrix}$$

from which we find that the first three vectors are linearly independent, hence form a basis. But that seems to be a bit ugly!

Another "method" which is easier, but might not work: find the three easiest looking vectors, try to see if they have any naive linear dependence, and then to verify that they are linearly independent, taken the determinant of the 3×3 matrix whose columns are those vectors. For example, the first, second, and last vectors look easiest, and don't obviously seem dependent. We try taking the determinant:

$$\det \begin{pmatrix} 1 & -3 & 2 \\ 4 & -5 & -3 \\ -2 & 8 & 1 \end{pmatrix} = -5 + (-18) + 64 - 20 - (-24) - (-12) = 57$$

which is nonzero, verifying that these vectors are linearly independent, hence form a basis of $\mathbb{R}^3$.

For the second part, we can use the procedure outlined on page 193–194 (Section 3.4, Example 4). We need to partially row reduce the matrix whose columns are the given vectors followed by a known basis (e.g., the standard basis) of the vector space. Then the four pivots will give four linearly independent vectors in $\mathbb{R}^4$ (of which two will be the columns with the given vectors and two more will be certain elements of the basis). The row reduction starts out:

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{pmatrix}$$

form which we see that there will be pivots in the first four columns, showing that $(2, 1, 3, 4), (1, 1, 1, 3), (1, 0, 0, 0), (0, 1, 0, 0)$ will be linearly independent, hence a basis of $\mathbb{R}^4$.

A tricky way to do this is to imagine putting these two vectors in a 4×4 matrix with nonzero determinant:

$$\begin{pmatrix} ? & ? & 2 & 1 \\ ? & ? & 1 & 1 \\ ? & ? & 3 & 1 \\ ? & ? & 4 & 3 \end{pmatrix}$$

What if we were lucky, and we could do this with a 2×2 block of zeros in the lower left:

$$\begin{pmatrix} ? & ? & 2 & 1 \\ ? & ? & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 3 \end{pmatrix}$$

Then by Exercise 21 from Section 4.3, the determinant of this matrix would be the determinant of the upper left 2×2 block times the determinant of the lower right block (which is 2). So, for example:

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 4 & 3 \end{pmatrix}$$

would work, showing again that adding the first two standard basis vectors on to our given two yields a basis. You see that this only works if the lower 2×2 determinant of the given vectors (thought of as the columns of a 4×2 matrix) is nonzero. If this happened to be zero, you could look at the upper 2×2 block for a nonzero determinant, which in this case also works, yielding the possible matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{pmatrix}$$

with nonzero determinant (think the transpose of the previously cited exercise), showing that the 3rd and 4th standard basis vectors could also be taken together with the given vectors to form a basis.

6. *Find the determinants of the following matrices with real entries:*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 7 & 9 & 11 & 13 \\ 0 & 0 & 6 & 7 & 8 & 9 \\ 0 & 0 & 8 & 9 & 10 & 11 \\ 0 & 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

To compute $\det A$, we could expand along the first column, i.e., write

$$\det A = \sum_{k=1}^{6} (-1)^{1+k} a_{k1} \det(\tilde{A}_{k1}).$$

Noticing that the top two rows of $\tilde{A}_{k1}$ are identical for $3 \leq k \leq 6$, so their determinants are zero, and that $a_{21} = 0$, we see that the only term that contributes is $a_{11} \det(\tilde{A}_{11})$. But

$$\tilde{A}_{11} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

is the 5×5 version of the matrix A. So continuing in this way with the same argument (i.e., we implicitly use induction, but you don't have to say that), we see that

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} = \cdots = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1.$$

Another way to do this is to row reduce:

$$\begin{pmatrix}1&1&1&1&1&1\\0&1&1&1&1&1\\1&0&1&1&1&1\\1&1&0&1&1&1\\1&1&1&0&1&1\\1&1&1&1&0&1\end{pmatrix} \to \begin{pmatrix}1&1&1&1&1&1\\0&1&1&1&1&1\\0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\end{pmatrix} \to \begin{pmatrix}1&1&1&1&1&1\\0&1&1&1&1&1\\0&0&1&1&1&1\\0&0&-1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\end{pmatrix} \to \cdots$$

to successively get an upper triangular matrix with all 1's on the diagonal.

To compute $\det B$, we notice that it is a block matrix of the form $\begin{pmatrix}X&Y&C\\0&D&E\\0&0&F\end{pmatrix}$ where each of these are 2×2 matrices. A generalization of Exercise 21 from Section 4.3 should be that such a matrix has determinant $\det(X)\det(D)\det(F)$. This can be proved by simply considering this matrix as a 2×2 matrix of blocks $\begin{pmatrix}A&(X|C)\\0&G\end{pmatrix}$ where $G=\begin{pmatrix}D&E\\0&F\end{pmatrix}$, we see that $\det(B)=\det(A)\det(G)=\det(A)\det(D)\det(F)$ by two applications of the aforementioned exercise. Therefore the determinant is $(-1)(-2)(4)=8$.