Yale University Department of Mathematics
**Math 225 Linear Algebra and Matrix Theory**
Spring 2014

Problem Set # 4 (due 4 pm Wednesday 19 February 2014)

**Notation:** Let $F$ be a field. Recall that $M_{m\times n}(F)$ is the $F$-vector space of $m \times n$ matrices with entries in $F$. For $A \in M_{m\times n}(F)$ define the **transpose** $A^t \in M_{n\times m}(F)$ as the matrix obtained by interchanging the rows and columns of $A$, i.e., the $ij$th entry of $A^t$ is the $ji$th entry of $A$.

For $A \in M_{n\times n}(F)$ a square matrix, define the **trace** $\mathrm{tr}(A) \in F$ to be the sum of the diagonal entries of $A$, i.e., if $A = (a_{ij})$ then $\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$. You can check that $\mathrm{tr} : M_{n\times n}(F) \to F$ is actually a linear map of $F$-vector spaces.

Given a linear map $T : V \to V$ between an $F$-vector space $V$ and itself, a subspace $W \subset V$ is called $T$**-invariant** if $R(T) \subset W$, i.e., $T$ takes vectors from $W$ back to vectors in $W$. For example, $\{0\} \subset V$ and $V \subset V$ are always $T$-invariant subspaces for any linear map $T : V \to V$.

Let $W_1$ and $W_2$ be vector spaces over a field $F$. Define the **cartesian product** $W_1 \times W_2$ to be the set of ordered pairs $(w_1, w_2)$ of vectors $w_1 \in W_1$ and $w_2 \in W_2$. For example, $F^2 = F \times F$.

Let $W_1 \subset V$ and $W_2 \subset V$ be subspaces of a vector space $V$. The notation $W_1 + W_2$ denotes the subspace of $V$ consisting of vectors $w_1 + w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$. Rephrasing it another way, $W_1 + W_2 = \mathrm{span}(W_1 \cup W_2)$. We say that $V$ is the **direct sum** of $W_1$ and $W_2$, and write $V = W_1 \oplus W_2$, if $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$.

**Reading:** FIS 2.2, 2.3, 2.4

**Problems:**

**1.** FIS 2.2 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 2bce, 4, 5acdfg, 8, 9, 11, 13.

**2.** FIS 2.3 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 4acd, 9, 11, 12 (If true, prove it, it false then provide a counterexample).

**3.** Let $W_1 \subset V$ and $W_2 \subset V$ be subspaces of an $F$-vector space $V$. Consider the map

$$\Sigma : W_1 \times W_2 \to W_1 + W_2$$

defined by $\Sigma(v, w) = v + w$. Verify that $\Sigma$ is a linear map and is onto. Prove that if $V = W_1 \oplus W_2$ then in fact $\Sigma$ is an isomorphism.

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