Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Spring 2012

Problem Set # 4 (Fri 17 Feb 2012) Selected Solutions

1. Let P be a point in $\mathbb{R}^3$ and let $\vec{v}$ be a direction vector at P. Find a parameterization of the line through P in the direction $\vec{v}$ and with constant speed 1.

Solution. If you just parameterize the line using the standard method (we have a point and a direction vector), then you get

$$\gamma(t) = P + t\vec{v}.$$

But computing the velocity of this parameterization, you find $\vec{\gamma}'(t) = \vec{v}$ for every t. This has speed $\|\vec{v}\|$, which may not be equal to 1, depending on which $\vec{v}$ you are given. So the real question is: can you find a direction vector pointing in the same direction as $\vec{v}$ (so that the line parameterized with be the same line) but having length 1? Yes. This is in the book on page 692. Given any nonzero vector $\vec{v}$, then the vector $\frac{1}{\|\vec{v}\|}\vec{v}$ points in the same direction as $\vec{v}$ (since they are proportional) and has length 1. Then use that to parameterize.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = 9 - 2x - 3y$. Let $P = (1, 2, 1)$.

Solution.

a) For each angle θ, the direction vector $\cos(\theta)\vec{\imath}+\sin(\theta)\vec{\jmath}$ is a unit vector (since $\sin^2 + \cos^2 = 1$) pointing in a direction making angle θ with the x-axis. So using this vector at the point $(1, 2)$ gives a parameterization

$$\gamma(t) = (1 + \cos(\theta)t, 2 + \sin(\theta)t)$$

of the desired line with constant speed 1.

b) The lift is

$$\alpha_\theta(t) = (1+\cos(\theta)t, 2+\sin(\theta)t, 9-2(1+\cos(\theta)t)-3(2+\sin(\theta)t)) = (1+\cos(\theta)t, 2+\sin(\theta)t, 1-2\cos(\theta)t-3\sin(\theta)t).$$

c) First we write down δ:

$$\delta(\theta) = \alpha_\theta(1) = (1 + \cos(\theta), 2 + \sin(\theta), 1 - 2\cos(\theta) - 3\sin(\theta))$$

and realize that if we parameterize the standard unit circle around $(1, 2)$ as

$$u(\theta) = (1 + \cos(\theta), 2 + \sin(\theta))$$

then $\delta(\theta)$ is actually the lift to the graph of f of the parameterized curve u. Now also we should realize that the graph of f has equation $z = f(x, y) = 9 - 2x - 3y$, and so it's a plane! Can you see why lifting a circle to a plane results in an ellipse?

d) The vector pointing in direction of greatest rate of change of f at $(1, 2)$ (i.e. greatest ascent on the graph of f at $(1, 2, 1)$) is given by the gradient

$$\nabla f|_{(1,2)} = -2\vec{\imath} - 3\vec{\jmath}.$$

To find the angle, we can use the inverse tangent function and draw a little triangle. We find that the angle this vector makes with the x-axis is $\tan(-3/-2) + \pi$.

e) First calculate the velocity (taking derivatives with respect to t)

$$\vec{\alpha}_\theta'(t) = \cos(\theta)\vec{\boldsymbol{\imath}} + \sin(\theta)\vec{\boldsymbol{\jmath}} - (2\cos(\theta) + 3\sin(\theta))\vec{\boldsymbol{k}}.$$

Since this doesn't depend on t any longer, evaluating at $t = 0$ gives

$$\vec{\alpha}_\theta'(1) = \cos(\theta)\vec{\boldsymbol{\imath}} + \sin(\theta)\vec{\boldsymbol{\jmath}} - (2\cos(\theta) + 3\sin(\theta))\vec{\boldsymbol{k}}.$$

f) The speed is

$$\|\vec{\alpha}_\theta'(t)\| = \sqrt{\cos^2(\theta) + \sin^2(\theta) + (2\cos(\theta) + 3\sin(\theta))^2} = \sqrt{1 + (2\cos(\theta) + 3\sin(\theta))^2}.$$

3. CM 17.1 Problem 48.

Solution.

a) The line $\gamma(t) = (2 + 3t, 5 + t, 2t)$ intersects the plane $x + y + z = 1$ when

$$(2 + 3t) + (5 + t) + 2t = 0$$

i.e. when $t = -7/6$. Then the point of intersection is $\gamma(-7/6) = (-3/2, 23/6, -7/3)$.

b) By vector, they mean "vector at the point of intersection." Otherwise, it's impossible! So we're looking for a vector that is both in the plane $x+y+z = 1$ and also perpendicular to the line $\gamma(t)$ at the point of intersection. But this last condition means exactly that the vector is perpendicular to the direction (which is $3\vec{\boldsymbol{\imath}}+\vec{\boldsymbol{\jmath}}+2\vec{\boldsymbol{k}}$) of the line. This means that the vector is in the plane normal to the vector $3\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + 2\vec{\boldsymbol{k}}$ through the point of intersection. So we are just looking for a vector *on the intersection of two planes* which we know how to do. Taking the cross product of the normal vectors of the planes will give you a vector pointing in the direction of the intersection at the point. This cross product is

$$(3\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + 2\vec{\boldsymbol{k}}) \times (\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}}) = \vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} - 2\vec{\boldsymbol{k}}.$$

So at the point $(-3/2, 23/6, -7/3)$, the vector $\vec{\boldsymbol{\imath}}+\vec{\boldsymbol{\jmath}}-2\vec{\boldsymbol{k}}$ points in the direction of this intersection (i.e. it is on the plane and perpendicular to the line). The endpoint of this vector $(-1/2, 29/6, -13/3)$ is then a point on this intersection.

c) Well, you are just now being asked to parameterize the line through the point $(-3/2, 23/6, -7/3)$ in the direction $\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} - 2\vec{\boldsymbol{k}}$. Fine.