Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Spring 2012

Problem Set # 1 (due Friday 27 Jan 2012) Solutions

1. *Graphs of Multivariable Functions*

a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function on $\mathbb{R}^2$. Give $\mathbb{R}^3$ the standard (x, y, z) coordinates. Describe the intersection of Γ_f with the x-y-plane by an implicit equation in terms of the function f. Draw this set for $f(x, y) = x^2 - xy$.

Solution.
The x-y-plane is given by the equation $z = 0$. The intersection of this plane with the graph Γ_f is given by the equation $f(x, y) = 0$, which is what we also called the *level set* of f at 0. When $f(x, y) = x^2 - xy$, this is the set of points in the plane (x, y) satisfying $x^2 - xy = 0$. Factoring, this is $x(x - y) = 0$. If a product of numbers is zero, then either one of them is, so either $x = 0$ or $x - y = 0$. Thus this is the (vertical) line $x = 0$ and the (diagonal) line $x = y$ in the x-y-plane.

b) Now let $f(x, y) = x^2 + y$. Draw the intersection of Γ_f with the x-z-plane and find both an implicit equation and a parameterization describing it.

Solution.
The x-z-plane is given by $y = 0$. The graph Γ_f has equation $z = f(x, y)$. So the intersection is the set of points (x, y, z) satisfying $z = f(x, 0) = x^2$ and $y = 0$. This looks like a parabola in the x-z-plane. A parameterization is $\gamma(t) = (t, 0, t^2)$.

3. CM Problems 12.2.24. Let S in $\mathbb{R}^3$ be the surface defined by $z = (x^2 + 1) \sin y + xy^2$.

Solution.

a) We want a "squared term" left when we intersect, so we could use the x^2 or the y^2 term. The problem is that the y^2 term will never be by itself without the $\sin y$ term (since $x^2 + 1$ is never 0). So we go for the x^2 term. So we want to intersect with some plane $y = \theta$. The intersection will then have the equation $z = x^2 \sin\theta + \sin\theta + x\theta^2$. To keep the x^2 term alive, we just want to ensure that $\sin\theta \neq 0$. For example, take $\theta = \pi/2$ (so that $\sin\theta = 1$), then the intersection is the parabola $z = x^2 + \frac{\pi^2}{4}x + 1$.

b) Now we want to kill the x^2 term and preserve the x term. So as above, we can choose $\theta = \pi$. The intersection is then the line $z = \pi x$ in the $y = \pi$ plane.

c) Now we want to preserve the sin term. Intersecting with the plane $x = 0$ does this nicely.

4. CM Exercise 12.3.14. Let $f(x, y) = 3x^2y + 7x + 20$.

Solution.

The level sets of f are the subsets in the x-y-plane given by $f(x, y) = c$ for some constant c. Since $f(5, 10) = 805$, the level set of f through the point $(5, 10)$ is implicitly defined by $f(x, y) = 805$. You can rewrite this as $3x^2y + yx - 785 = 0$.

But what on earth does this look like? For $y = 0$, for example, there is a unique solution, $x = 785/7$, so the level set is just the point $(785/7, 0)$. Now for a fixed $y \neq 0$, think of this as a quadratic equation in x. So this can have two, one, or no solutions for x, depending on what the quadratic formula gives:

$$x = \frac{-7 \pm \sqrt{49 - 9420y}}{6y}.$$

When the discriminant is zero (i.e. $y = 49/9420$) then $x = -1570/7$ is the unique solution, so again the level set is a point $(-1570/7, 49/9420)$. For $y \geq 49/9420$, there are no solutions for x, while when $y \leq 49/9420$, there are two solutions for x. Can you now imagine what the level set looks like? Try graphing it on your computer!

5. CM Problem 12.5.30. Describe the level surfaces of $f(x, y, z) = x^2 - y^2 + z^2$.

Solution.

The level surfaces of f are implicitly defined surfaces of the form $f(x, y, z) = c$ for $c \in \mathbb{R}$. If you look at the "Catalog of Surfaces" on page 671, you'll see that the level surface is a two-sheeted hyperboloid for $c < 0$, a cone for $c = 0$, and a one-sheeted hyperboloid for $c > 0$. This is called a *family of quadrics with degeneration*. A "quadric" is a higher dimensional analogue of a conic section. "Degeneration" refers to the fact that a cone is thought of as a "degenerate" quadric, just like two intersecting lines is a degenerate conic section. I study families of quadrics with degeneration in my research!

6. CM Problem 13.3.40. Write an equation of the plane parallel to the plane $2x + 4y - 3z = 1$ and through the point $(1, 0, -1)$.

Solution.

Two planes are parallel if they have proportional normal vectors. From the equation of the plane $2x+4y-3z = 1$, we see that $2\vec{\imath}+3\vec{\jmath}-3\vec{k}$ is a normal vector. So our plane has this normal vector and goes through the point $(1, 0, -1)$, so we can use the point/normal formula to get the equation $2x+4y-3z = 2\cdot 1+3\cdot 0+(-3)\cdot(-1) = 5$.

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