Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Spring 2012

Midterm # 1 (Tue 21 Feb 2012) Practice Exam Solution Guide

Practice problems: The following assortment of problems is inspired by what will appear on the midterm exam, but is not necessarily representative of the length of the midterm exam. On the actual midterm exam, you will have your choice of solving 6 problems out of 7 given.

1. Find an implicit equation for the graph of the function $f(x, y) = x^2 \sin(xy)$ in $\mathbb{R}^3$.

Solution. The graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ has the implicit equation $z = f(x, y)$. In our case, it will be $z = x^2 \sin(xy)$.

2. Consider the implicitly defined surface $\{2xyz + xy + z^2 + 2 = xz^2 + x + y + 2z\}$ in $\mathbb{R}^3$.

a) Find the points (there are two of them!) of intersection of the surface with the line through the origin in direction $\vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}}$.

b) Write equations (in the form ax+by+cz=e) for the tangent planes to the surface at the points of intersection.

c) Find the line of intersection of these two tangent planes.

Solution.

a) The line in question has parameterization $\gamma(t) = (0, t, t)$. Plugging this into the equation of the surface yields

$$2 \cdot 0 \cdot t \cdot t + 0 \cdot t + t^2 + 2 = 0 \cdot t^2 + 0 + t + 2t$$

which is the same as

$$t^2 - 3t + 2 = (t - 1)(t - 2) = 0$$

which only happens when $t = 1$ or $t = 2$. Thus the two points on intersection are $\gamma(1) = (0, 1, 1)$ and $\gamma(2) = (0, 2, 2)$.

b) We view S as the level surface of the function

$$F(x, y, z) = 2xyz + xy + z^2 + 2 - xz^2 - x - y - 2z$$

so that a normal vector to any point (a, b, c) on S is given by the gradient

$$\vec{n} = \nabla F|_{(a,b,c)} = (2bc + b - c^2 - 1)\vec{\boldsymbol{\imath}} + (2ac + a - 1)\vec{\boldsymbol{\jmath}} + (2ab + 2c - 2ac - 2)\vec{\boldsymbol{k}}.$$

At the two points $\gamma(1)$ and $\gamma(2)$, these normal vectors are given by

$$\vec{n}_1 = \nabla F|_{(0,1,1)} = \vec{\boldsymbol{\imath}} - \vec{\boldsymbol{\jmath}}, \qquad \vec{n}_2 = \nabla F|_{(0,2,2)} = 5\vec{\boldsymbol{\imath}} - \vec{\boldsymbol{\jmath}} + 2\vec{\boldsymbol{k}},$$

and so the tangent planes are given by $\vec{n}_i \cdot \vec{X} = \vec{n}_i \cdot \gamma(i)$ for $i = 1, 2$, which we now write

$$x - y = -1, \qquad 5x - y + 2z = 2,$$

respectively.

c) To find a point on the intersection of the two planes, we can try to see where the intersection crosses the y-z-plane. So setting $x = 0$ yields $y = 1$ (from the first equation) and then $z = 2/3$ (from the second equation). Finally, $(x, y, z) = (0, 1, 3/2)$ is a point on the intersection of the two planes.

Now one method is to find another point. If you like this, we can try to see where the intersection crosses the x-z-plane. So setting $y = 0$ yields $x = -1$ (in the first equation) and then $z = 7/2$ (from the second equation). Finally, $(x, y, z) = (-1, 0, 7/2)$ is another point. The vector pointing from the first to the second point is $-\vec{\imath} - \vec{\jmath} + 2\vec{k}$ and so the line connecting the two points has parameterization

$$\gamma(t) = (-t, 1 - t, \frac{3}{2} + 2t).$$

The other method is to take the cross product of the normal vectors to get a vector pointing along the intersection. To this end, we compute

$$\vec{n}_1 \times \vec{n}_2 = -2\vec{\imath} - 2\vec{\jmath} + 4\vec{k}$$

and then parameterize the line from $(0, 1, 3/2)$ pointing in this direction

$$\gamma(t) = (-2t, 1 - 2t, \frac{3}{2} + 4t).$$

These are two parameterizations (with different speeds) for the same line. Which do you like the best?

3. Let S be the surface $\{z^2 = x^2 + y^2\}$ in $\mathbb{R}^3$. For each of the following intersections, choose the shape of that best describes it:

a) The intersection of S with the plane $z = 0$.	*a)* Circle
b) The intersection of S with the plane $z = -1$.	*b)* Ellipse
c) The intersection of S with the plane $y = 0$.	*c)* Hyperbola
d) The intersection of S with the plane $y = -1$.	*d)* Parabola
e) The intersection of S with the plane $x - z = 0$.	*e)* Point
f) The intersection of S with the plane $x - z = -1$.	*f)* Line
g) The intersection of S with the plane $x + 2z = 0$.	*g)* Two parallel lines
h) The intersection of S with the plane $x + 2z = -1$.	*h)* Two intersecting lines

Solution. *a)-e), b)-a), c)-h), d)-c), e)-f), f)-d), g)-e), h)-b).*

4. Let $P = (1, 2, 3)$ and $Q = (2, 3, 4)$ be points in $\mathbb{R}^3$ and let $\vec{v} = \vec{\imath} - \vec{\jmath} + 2\vec{k}$ be a direction vector at P. Find an equation for each of the following planes:

a) Through P with normal $\vec{v}$.
b) Through P, Q, and the endpoint of $\vec{v}$.
c) Through P and containing the direction vectors $\vec{v}$ and $\overrightarrow{PQ}$.
d) Through Q and $P + Q$ and parallel to $\vec{v}$.

Also, find the volume of the tetrahedral shape spanned by $\vec{v}$, $\overrightarrow{PQ}$, and $\overrightarrow{PO}$ at P, where O is the origin.

Solution.

a) Using the point/normal formula with P and $\vec{v}$ you get $x - y + 2z = 5$.
b) The endpoint of $\vec{v}$ is the point $P + \vec{v} = (2, 1, 5)$, which we'll call E. Now we have three points P, Q, and E, so the cross product $\overrightarrow{PQ} \times \overrightarrow{PE}$ gives a normal at P. But since $\overrightarrow{PE} = \vec{v}$ by definition, save some time and realize that $\overrightarrow{PQ} \times \vec{v}$ gives a normal at P, which is $3\vec{\imath} - \vec{\jmath} - 2\vec{k}$. Then the plane is $3x - y - 2k = -5$.
c) This is the same plane as in *b)*!
d) Thinking of Q as the base point, a plane parallel to $\vec{v}$ means that $\vec{v}$, thought of as a direction vector at Q, is contained on the plane. Then $\overrightarrow{Q(P + Q)} = \vec{\imath} + 2\vec{\jmath} + 3\vec{k}$ is also on this plane

so we can find a normal by taking the cross product, $(\vec{\imath}+2\vec{\jmath}+3\vec{k})\times\vec{v} = 7\vec{\imath}+\vec{\jmath}-3\vec{k}$. Then the plane is $7x+y-3z=5$.

Finally, the tetrahedral shape is half of the parallelepiped spanned by three vectors, so its volume is $\frac{1}{2}|\vec{v}\cdot(\overrightarrow{PQ}\times\overrightarrow{PO})| = 5/2$, as explained in the book.

5. For the plane $2x-3y+z=6$ find a point on the plane and a direction vector at that point normal to the plane.

Solution. For example, the point $(0,0,6)$ is on the plane, and as usual, $2\vec{\imath}-3\vec{\jmath}+\vec{k}$ is normal to the plane at any point.

6. Find an equation of the tangent plane to each of the following surfaces in $\mathbb{R}^3$ at the given points:

a) The graph of $f(x,y)=xy2^{xy}$ at the point above $(1,2)$.
b) The implicit surface $xy^2+yz^2+zx^2=3$ at the point $(1,1,1)$.

Solution.

a) The point on the graph is $(1,2,f(1,2))=(1,2,8)$ and a normal to the tangent plane at this point is $\nabla f|_{(1,2)}-\vec{k} = 8(1+2\ln(2))\vec{\imath}+4(1+2\ln(2))\vec{\jmath}-\vec{k}$. Use the point/normal equation.
b) This surface a level surface of the function $F(x,y,z)=xy^2+yz^2+zx^2-3$, so a normal to the tangent plane at $(1,1,1)$ is $\nabla F|_{(1,1,1)} = 3\vec{\imath}+3\vec{\jmath}+3\vec{k}$. Use point/normal equation.

7. Let S be the surface $4x^2+y^2+z^2=36$ in $\mathbb{R}^3$ and let $\vec{v}=\vec{\imath}+\vec{\jmath}+\vec{k}$ be a direction vector.

a) Find all points on the intersection of S with the line through $(\frac{1}{2},1,1)$ in the direction $\vec{v}$.
b) Find all points on the simultaneous intersection of S, the plane $\{8x+y-z=0\}$, and the plane $\{x-y+z=9\}$.
c) Find all points on S whose tangent plane is normal to $\vec{v}$.

Solution.

a) This line is parameterized by $\gamma(t) = (\frac{1}{2}+t, 1+t, 1+t)$. The intersection points of this line and S are at times t when $4(\frac{1}{2}+t)^2+(1+t)^2+(1+t)^2=36$. Simplifying, we have $6t^2+8t-33=0$, giving us two values of t, by the quadratic equation

$$t = \frac{-8\pm\sqrt{8^2-4\cdot 6\cdot(-33)}}{12} = \frac{-4\pm\sqrt{214}}{6}$$

so that $\gamma(t)$ intersects S at the points

$$\gamma((-4\pm\sqrt{214})/6) = ((-1\pm\sqrt{214})/6, (2\pm\sqrt{214})/6, (2\pm\sqrt{214})/6).$$

b) First we'll parameterize the intersection of the planes, which is a line. To this end, we first need to find a point on their intersection, i.e. we're solving a system of two linear equations. Use whatever method is your favorite. I found the point $(1,0,8)$. Now at that point, normals to the two planes are given by $8\vec{\imath}+\vec{\jmath}-\vec{k}$ and $\vec{\imath}-\vec{\jmath}+\vec{k}$, then their cross product $-9\vec{\jmath}-9\vec{k}$ is pointing in the direction of the intersection line. The scaled vector $\vec{\jmath}+\vec{k}$ is also pointing in this direction. So a parameterization of the intersection of planes is $\gamma(t)=(1,t,8+t)$. Now find the intersection of this line with S as before by substituting $4\cdot 1^2+t^2+(8+t)^2=36$, which simplifies to $(t+4)^2=0$, so the only intersection is when $t=4$, i.e. at the point $\gamma(4)=(1,-4,4)$.

c) Since S is the level surface of the function $F(x,y,z) = 4x^2 + y^2 + z^2 - 36$, the general point $P = (x,y,z)$ on S has tangent plane normal to $\nabla F|_P = 8x\vec{\boldsymbol{\imath}} + 2y\vec{\boldsymbol{\jmath}} + 2z\vec{\boldsymbol{k}}$. So P will have tangent plane normal to $\vec{v}$ when $\nabla F|_P$ is a scalar multiple of $\vec{v}$, i.e. $\nabla F|_P = \lambda\vec{v}$ for some $\lambda \in \mathbb{R}$. This turns into three equations $8x = 2y = 2z = \lambda$, which only happens at point of the form $P = (\lambda, 4\lambda, 4\lambda)$. But now which such points are actually on S? Substituting we arrive at $4\lambda^2 + (4\lambda)^2 + (4\lambda)^2 = 36$, which simplifies to $\lambda^2 = 1$, i.e. at $\lambda = \pm 1$. So the points of simultaneous intersection are $(1,4,4)$ and $(-1,-4,-4)$.

8. Let $f(x,y) = y - x^2$, let Γ be its graph in $\mathbb{R}^3$, and let P be the point on Γ above $(2,3)$.

a) Parameterize the line of steepest ascent on Γ at P, i.e. the line you will start to follow going up the hill starting at P in the direction of steepest ascent.

b) Parameterize the line of no ascent or descent on Γ at P, i.e. the line tangent to the contour of Γ through P.

c) Parameterize the contour line of Γ through P, i.e. find a parameterized curve whose image is this contour.

Solution.

a) First, $P = (2,3,f(2,3)) = (2,3,-1)$. The direction (in the x-y-plane) of greatest increase of f is $\nabla f_{(2,3)} = -4\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}}$. The nice formula from class stated that the vector

$$\vec{v}_{f,(2,3)} = \nabla f|_{(2,3)} + \|\nabla f|_{(2,3)}\|^2\vec{\boldsymbol{k}} = -4\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + 17\vec{\boldsymbol{k}}$$

is pointing toward the steepest ascent of Γ at P. So parameterizing the line through P in the direction $\vec{v}_{f,(2,3)}$ yields

$$\gamma(t) = (2 - 4t, 3 + t, -1 + 17t).$$

b) Here's a geometric way of finding this. We've seen in class that on the tangent plane, the direction of greatest ascent is perpendicular to the direction of no ascent (the direction tangent to the level curve through that point). How to find a direction vector perpendicular to a certain direction vector on a random plane? Well, we know a normal to the tangent plane, namely $\vec{n} = \nabla f|_{(2,3)} - \vec{\boldsymbol{k}}$. By the properties of the cross product, for any $\vec{v}$ on the tangent plane, $\vec{n} \times \vec{v}$ will again be on the tangent plane and also *perpendicular* to $\vec{v}$. So we can just calculate $\vec{n} \times \vec{v}_{f,(2,3)} = (-4\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} - \vec{\boldsymbol{k}}) \times (-4\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + 17\vec{\boldsymbol{k}}) = 18\vec{\boldsymbol{\imath}} + 72\vec{\boldsymbol{\jmath}}$. Then a parameterization of the line is $(2 + 18t, 3 + 72t, -1)$. You can also look at problem CM 14.3.32a for an alternate method.

c) Since $f(2,3) = -1$, the contour of Γ through P is the intersection of Γ with the plane $z = -1$, i.e. $y - x^2 = -1$ and $z = -1$. Thus solving for $y = x^2 - 1$, we can parameterize this contour by $\gamma(t) = (t, t^2 - 1, -1)$.

Since γ parameterizes the contour line, and goes through P at $t = 2$, we can again check part *b)* by computing the velocity vector $\gamma'(2) = \vec{\boldsymbol{\imath}} + 4\vec{\boldsymbol{\jmath}}$, which will be tangent to level curve. Indeed, $\gamma'(2)$ is a scalar multiple of $18\vec{\boldsymbol{\imath}} + 72\vec{\boldsymbol{\jmath}}$ from *b)*, so they define the same line.

9. Let $\gamma(t) = (t, t^2, t^3)$ be a parameterized curve in $\mathbb{R}^3$.

a) Calculate the velocity $\gamma'(t)$.

b) Find all times t when $\gamma(t)$ is moving in a direction normal to $\vec{\boldsymbol{\imath}} - \vec{\boldsymbol{\jmath}} - \vec{\boldsymbol{k}}$.

c) Find all times t when $\gamma(t)$ is moving at speed 1.

d) Find all times t when $\gamma(t)$ intersects the plane $3x - 4y + z = 0$.

e) Find a plane that never intersects $\gamma(t)$.

Solution.

a) $\gamma'(t) = \vec{\boldsymbol{\imath}} + 2t\vec{\boldsymbol{\jmath}} + 3t^2\vec{\boldsymbol{k}}$.

b) We want to find times t when $\gamma'(t) \cdot (\vec{\boldsymbol{\imath}} - \vec{\boldsymbol{\jmath}} - \vec{\boldsymbol{k}}) = 1 - 2t - 3t^2 = 0$, i.e.

$$t = (2 \pm \sqrt{2^2 - 4 \cdot (-3)})/2 = (2 \pm 4)/(2 \cdot (-3)) = -1,\ \frac{1}{3}$$

by the quadratic formula.

c) We want to find times t when $\|\gamma'(t)\| = \sqrt{1^2 + 4t^2 + 9t^4} = 1$. This happens when $1 + 4t^2 + 9t^4 = 1$. Simplifying gives $t^2(4 + 9t^2) = 0$. But this is only possible when $t = 0$.

d) We want to find times t when $3t - 4t^2 + t^3 = 0$. Simplifying gives $t(3 - 4t + t^2) = t(t-1)(t-3) = 0$, i.e. when $t = 0,\ 1,\ 3$.

e) Given the general plane $ax + by + cz = d$, $\gamma(t)$ intersects it when $at + bt^2 + ct^3 = d$. We'd like to find a choice of a, b, c, d so that this is never solvable. First note that if $c \neq 0$, then this equation is a cubic, which is *always* solvable (if you don't see why, you should think about this)! So $\gamma(t)$ passes through any plane with $c \neq 0$! This gives you a sense of how "twisted" the curve γ is. So we definitely want $c = 0$. Then we're left with $bt^2 + at - d = 0$, and by the quadratic equation, this is not solvable if the discriminant (the part under the square root sign) is negative. In this case, the discriminant is $a^2 + 4bd$, so for example, if $a = b = 1$ and $d = -1$, then the discriminant is negative. We conclude that $\gamma(t)$ never intersects the plane $x + y = -1$, for example.