Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Spring 2012

Midterm # 1 (Tue 21 Feb 2012) Practice Exam

Directions: You will have 1 hour and 10 minutes for the midterm exam. No electronic devices will be allowed. No notes will be allowed. Should you need them, you will have the following formulas during the exam:

Dot/cross product: Let $\vec{v} = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$ and $\vec{w} = w_1\vec{\imath} + w_2\vec{\jmath} + w_3\vec{k}$ be direction vectors at a point $P \in \mathbb{R}^3$ with angle θ between them.

$$\vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3, \qquad \|v\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$\vec{v}\times\vec{w} = (v_2w_3 - v_3w_2)\vec{\imath} + (v_3w_1 - v_1w_3)\vec{\jmath} + (v_1w_2 - v_2w_1)\vec{k}$$

$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta, \qquad \|\vec{v}\times\vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin\theta$$

Planes: If $\vec{X} = (x, y, z)$ is the position vector of the general point $(x, y, z) \in \mathbb{R}^3$, then the plane through a point P with normal $\vec{n}$ has the equation $\vec{n}\cdot\overrightarrow{PX} = 0$.

Quadratic equation: the (complex) solutions of $az^2 + bz + c = 0$ are

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Text material covered: CM 12.1–5, 13.1–4, 14.1–5, 17.1–2.

Practice problems: The following assortment of problems is inspired by what will appear on the midterm exam, but is not necessarily representative of the length of the midterm exam. On the actual midterm exam, you will have your choice of solving 6 problems out of 7 given.

1. Write an implicit equation for the graph Γ_f of the function $f(x, y) = x^2\sin(xy)$.

2. Consider the implicitly defined surface $\{2xyz + xy + z^2 + 2 = xz^2 + x + y + 2z\}$ in $\mathbb{R}^3$.

a) Find the points (there are two of them!) of intersection of the surface with the line through the origin in direction $\vec{\jmath} + \vec{k}$.
b) Write equations (in the form $ax + by + cz = e$) for the tangent planes to the surface at the points of intersection.
c) Write a parameterization of the line of intersection of these two tangent planes.

3. Let S be the surface $\{z^2 = x^2 + y^2\}$ in $\mathbb{R}^3$. For each of the following intersections, choose the shape of that best describes it:

a) The intersection of S with the plane $z = 0$.
b) The intersection of S with the plane $z = -1$.
c) The intersection of S with the plane $y = 0$.
d) The intersection of S with the plane $y = -1$.
e) The intersection of S with the plane $x - z = 0$.
f) The intersection of S with the plane $x - z = -1$.
g) The intersection of S with the plane $x + 2z = 0$.
h) The intersection of S with the plane $x + 2z = -1$.

a) Circle
b) Ellipse
c) Hyperbola
d) Parabola
e) Point
f) Line
g) Two parallel lines
h) Two intersecting lines

4. Let $P = (1, 2, 3)$ and $Q = (2, 3, 4)$ be points in $\mathbb{R}^3$ and let $\vec{v} = \vec{\boldsymbol{\imath}} - \vec{\boldsymbol{\jmath}} + 2\vec{\boldsymbol{k}}$ be a direction vector at P. Find an equation for each of the following planes:

a) Through P with normal $\vec{v}$.
b) Through P, Q, and the endpoint of $\vec{v}$.
c) Through P and containing the direction vectors $\vec{v}$ and $\overrightarrow{PQ}$.
d) Through Q and $P + Q$ and parallel to $\vec{v}$.

Also, find the volume of the tetrahedral shape spanned by $\vec{v}$, $\overrightarrow{PQ}$, and $\overrightarrow{PO}$ at P, where O is the origin.

5. For the plane $\{2x - 3y + z = 6\}$ find a point on the plane and write a direction vector at that point normal to the plane.

6. Find an equation of the tangent plane to each of the following surfaces in $\mathbb{R}^3$ at the given points:

a) The graph of $f(x, y) = xy2^{xy}$ at the point above $(1, 2)$.
b) The implicit surface $\{xy^2 + yz^2 + zx^2 = 3\}$ at the point $(1, 1, 1)$.

7. Let S be the surface $\{4x^2 + y^2 + z^2 = 36\}$ in $\mathbb{R}^3$ and let $\vec{v} = \vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}}$ be a direction vector.

a) Find all points on the intersection of S with the line through $(\frac{1}{2}, 1, 1)$ in the direction $\vec{v}$.
b) Find all points on the simultaneous intersection of S, the plane $\{8x + y - z = 0\}$, and the plane $\{x - y + z = 9\}$.
c) Find all points on S whose tangent plane is normal to $\vec{v}$.

8. Let $f(x, y) = y - x^2$, let Γ_f be its graph in $\mathbb{R}^3$, and let P be the point on Γ_f above $(2, 3)$.

a) Parameterize the line of steepest ascent on Γ_f at P, i.e. the line you will start to follow going up the hill starting at P in the direction of steepest ascent.
b) Parameterize the line of no ascent or descent on Γ_f at P, i.e. the line you will start to follow going in the direction of no ascent/descent.
c) Parameterize the contour of f that passes through the point P, i.e. find a parameterized curve whose image is this contour.

9. Let $\gamma(t) = (t, t^2, t^3)$ be a parameterized curve in $\mathbb{R}^3$.

a) Calculate the velocity $\gamma'(t)$.
b) Find all times t when $\gamma(t)$ is moving in a direction normal to $\vec{\boldsymbol{\imath}} - \vec{\boldsymbol{\jmath}} - \vec{\boldsymbol{k}}$.
c) Find all times t when $\gamma(t)$ is moving at speed 1.
d) Find all times t when $\gamma(t)$ intersects the plane $3x - 4y + z = 0$.
e) Find a plane that never intersects $\gamma(t)$.