Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Spring 2012

Final Problem Set # 11 (due Tuesday May 01 2012)

1. CM 16.7 Exercise 2, 4, 6
Problems 16, 17, 20 (afterward switch to polar coordinates), 21, 26

2. (Extra Credit) Let R_α be the region bounded by two cylinders of radius 1 whose axes intersect with an angle α between them. Assume $0 < \alpha \leq \pi/2$. Feel free to orient one of the cylinders in a convenient position (like along the z-axis, or perhaps along the y-axis, for example). Find the volume of R_α as a function of α. You already know from class that the volume of $R_{\pi/2}$ is $16/3$. You may find that making a change of coordinates will help you! What happens to the volume of R_α as $\alpha \to 0$?

Reading: CM 17.5, 19.1–3

3. CM 19.1 Exercises 6, 10, 32

4. CM 19.3 Exercise 2, 3

Parameterized surfaces and flux. Let R be a region in $\mathbb{R}^2$, in which we use the coordinates (s,t). A **parameterized surface** in $\mathbb{R}^3$ is a mapping $\varphi : R \to \mathbb{R}^3$. The image S of φ is a **surface** in $\mathbb{R}^3$.

In analogy with the velocity vector of a parameterized curve, a parameterized surface $\varphi : R \to \mathbb{R}^3$ has two different **partial derivative vectors**. Expanding out into coordinate functions $\varphi(s,t) = (\varphi_1(s,t), \varphi_2(s,t), \varphi_3(s,t))$, then define

$$\frac{\partial \varphi}{\partial s} = \frac{\partial \varphi_1}{\partial s}\vec{\boldsymbol{\imath}} + \frac{\partial \varphi_2}{\partial s}\vec{\boldsymbol{\jmath}} + \frac{\partial \varphi_3}{\partial s}\vec{\boldsymbol{k}}, \qquad \frac{\partial \varphi}{\partial t} = \frac{\partial \varphi_1}{\partial t}\vec{\boldsymbol{\imath}} + \frac{\partial \varphi_2}{\partial t}\vec{\boldsymbol{\jmath}} + \frac{\partial \varphi_3}{\partial t}\vec{\boldsymbol{k}}.$$

Then $\frac{\partial \varphi}{\partial s}$ and $\frac{\partial \varphi}{\partial t}$ are always tangent vectors to the surface. Their cross product $\frac{\partial \varphi}{\partial s} \times \frac{\partial \varphi}{\partial t}$ is then always a normal vector to the surface (or zero in some bad cases).

Let $\vec{F}$ be a vector field on $\mathbb{R}^3$, and S a surface in $\mathbb{R}^3$ with parameterization $\varphi : R \to R^3$. Then the **flux integral** of $\vec{F}$ over S is

$$\int_S \vec{F} = \int_R (\vec{F} \circ \varphi) \cdot \left(\frac{\partial \varphi}{\partial s} \times \frac{\partial \varphi}{\partial t} \right).$$

You should think of a flux integral as a 2-dimensional version of a line integral.

For example, if $R = \{\, (s,t) \,:\, a \leq s \leq b,\ c \leq t \leq d \,\}$ is a box, then

$$\int_S \vec{F} = \int_{t=c}^{d} \int_{s=a}^{b} \vec{F}(\varphi(s,t)) \cdot \left(\frac{\partial \varphi}{\partial s}\Big|_{(s,t)} \times \frac{\partial \varphi}{\partial t}\Big|_{(s,t)} \right) ds dt.$$

This should remind you of the formula for a line integral given a parameterized curve.

Note that in CM, they write $\vec{r}$ for a parameterized surface (what we're calling φ) and $\int_S \vec{F}\, d\vec{A}$ for the flux integral.

Emory University, Department of Mathematics & CS, 400 Dowman Dr NE W401, Atlanta, GA 30322

E-mail address: auel@mathcs.emory.edu