Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Spring 2012

Final (Tue 08 May 2012, 08:30–11:00 am, MSC W301) Study Guide and Practice Exam

Directions: You will have 2 hour and 30 minutes for the final exam. No electronic devices will be allowed. One page ($8.5'' \times 11''$ front and back) of prepared notes will be allowed. You must show your work on all problems.

Topics/chapters to study:

Section I

a) real line $\mathbb{R}$, Cartesian plane $\mathbb{R}^2$, 3-space $\mathbb{R}^3$, n-space $\mathbb{R}^n$, points $P = (x, y, z)$ in $\mathbb{R}^3$, displacement vectors $\vec{v} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$ at a point, displacement vector between points $\overrightarrow{PQ}$, magnitude $\|\vec{v}\|$, dot product $\vec{v}\cdot\vec{w}$ (angle formula), normal vector $\vec{n}$ to a plane, equation of a plane $\vec{n}\cdot\overrightarrow{PX} = 0$ with normal $\vec{n}$ through P, cross product $\vec{v}\times\vec{w}$ (angle formula), planes through three points, parameterizing the line that is the intersection of two planes. HW 1-2 info parts, CM 12.1, 13.1–4.

b) Functions $f : \mathbb{R}^n \to \mathbb{R}$ (mostly $n = 1, 2, 3$), graphs Γ_f of functions $z = f(x, y)$ or $z = f(x, y, z)$, graphs of common functions (p. 671), contours/level curves $f(x, y) = c$ and level surfaces $f(x, y, z) = c$, intersecting graphs of functions with planes. CM 12.1–12.5.

c) Partial derivatives $\frac{\partial f}{\partial x}$, gradient ∇f and its properties (pointing to direction of greatest ascent, magnitude is the rate of increase), directional derivatives f_u, vector pointing toward greatest ascent $\nabla f + \|\nabla f\|^2\vec{k}$, tangent plane to graphs of functions $f : \mathbb{R}^2 \to \mathbb{R}$ has normal $\vec{n} = \nabla f - \vec{k}$, tangent plane to level surface of $f : \mathbb{R}^3 \to \mathbb{R}$ has normal $\vec{n} = \nabla f$. HW 3 info part, CM 14.1–5

d) Parameterized curves $\gamma : \mathbb{R} \to \mathbb{R}^n$ (for $n = 2, 3$), parameterized lines $\gamma(t) = \vec{P} + t\vec{v}$ for $0 \le t \le 1$ through point P in direction $\vec{v}$, parameterized circles $\gamma(t) = (\cos(t), \sin(t))$ for $0 \le t \le 2\pi$, velocity vector of a parameterized curve $\gamma'(t)$, lifting curves in the plane $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ to a graph $z = f(x, y)$, parameterizing the line of steepest ascent/descent and of no ascent/descent. HW 4 info part, CM 17.1–2.

Section II

a) Vector fields $\vec{F}$, constant vector fields, gradient fields $\vec{F} = \nabla f$, visualizing vector fields, regions of definition, flow curves $\vec{F}(\gamma(t)) = \gamma'(t)$. HW 5 info part, CM 17.3–4.

b) Line integrals $\int_\gamma \vec{F}$, computing line integrals via a parameterization $\int_a^b \vec{F}(\gamma(t))\cdot\gamma'(t)\,dt$, line integral is independent of parameterization chosen, Fundamental Theorem of Calculus for Line Integrals $\int_\gamma \nabla f = f(Q) - f(P)$ if γ goes from P to Q. HW 5–6 info part, CM 18.1–18.3.

c) Path-independent vector fields, gradient fields are path-independent (by FTC), tests for path-independence (two tests for path-independence: having a potential, zero scalar curl in a simply connected region of definition; two tests for non-path-independence: non-zero integral around a closed curve, non-zero scalar curl), finding potential functions of vector fields (solving differential equations), examples of vector fields (non-zero scalar curl, zero scalar curl but not path-independent, zero scalar curl on non-simply-connected regions that is path-independent). Midterm 2 practice solutions, CM 18.1–4.

d) Multivariable integrals $\int_R f$, iterated integrals/order of integration, double/triple integrals, Fubini's theorem on integration over boxes (can switch the order of iterated integration over boxes), calculating area and volume $\int_R 1$ (double, triple integrals), calculating volume under the graph of a 2-variable function $\int_R f$, integration in Cartesian (permutations of $dxdy$ or $dxdydz$), polar (permutations of $rdrd\theta$), and cylindrical coordinates (permutations of $rdrd\theta dz$). HW 7–9 info part, CM 16.1–3.

Section III

a) Integration in spherical coordinates (permutations of $\rho^2 \sin\phi \, d\rho d\theta d\phi$), general change of variables theorem

$$\int_{\Phi(R)} f(x,y)\,dxdy = \int_R f(x(s,t),y(s,t))\,|\det J_{(s,t)}\Phi|\,dudv,$$

jacobian matrix $J\Phi = \left(\frac{\partial(x,y)}{\partial(s,t)}\right)$, linear change of variables and matrices. CM 16.1–16.7.

b) Flux integrals: constant vector field over flat region ("constant vector field $\cdot$ area vector"), general formula for flux integral

$$\int_S \vec{F} = \int_R (\vec{F}\circ\varphi)\cdot\left(\frac{\partial\varphi}{\partial s}\times\frac{\partial\varphi}{\partial t}\right),$$

given a parameterization $\varphi : R \to \mathbb{R}^3$ of a surface S, parameterization of graphs. HW 11 info part, CM 17.5, 19.1–3.

c) Sometimes you can make your life easier by using Green's theorem to solve line integrals. Notes from last day of class, CM 18.4, 20.1–2.

Note: The final exam will consist of three section, each of approximately equal worth. Each section will consist of: one multiple choice/true-false question with up to 4 parts with short explanation, and two computational questions with up to 3 parts each where you will have to show your work. There will also be one additional problem which you can elect to replace any of the other problems.

Practice problems: The following assortment of problems is inspired by what will appear on the final exam, but is not necessarily representative of the length of the midterm exam.

Section I

1. Determine whether the following statements are always true, sometimes true/sometimes false, or always false. You do not need to justify your answer.

a) The surface implicitly defined by $z - y = h(x)$, where $h(x)$ is a function of x, is the graph of some function $f(x,y)$.

b) If $c \neq c'$ then the level curves $\{f(x,y) = c\}$ and $\{f(x,y) = c'\}$ never intersect.

c) Given a point $P \in \mathbb{R}^3$ and a plane S through P, there is a unique unit normal vector to S at P.

d) Given a point $P \in \mathbb{R}^3$, a plane S through P, and two vectors $\vec{v}$ and $\vec{w}$ on S with tail at P, then $\vec{v}\times\vec{w}$ is a normal vector to S at P.

e) Given vectors $\vec{v}$ and $\vec{w}$ at a point $P \in \mathbb{R}^3$, then $\vec{v}\times\vec{w} = \vec{w}\times\vec{v}$.

f) There is a function $f(x,y)$ with $\frac{\partial f}{\partial x} = y^2$ and $\frac{\partial f}{\partial y} = x^2$.

g) Given a function $f(x,y)$ and a point $P \in \mathbb{R}^2$, there is a direction $\vec{u}$ at P where the rate of change of f is 0.

2. Let $P = (1,2,3)$, $Q = (3,5,7)$, and $R = (2,5,3)$. Find an equation for the plane through P, Q, and R. Find a unit normal vector to this plane. Find the angle between $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Find the area of the triangle with vertices P, Q, and R. Find the shortest distance from R to the line through P and Q.

3. Finding tangent lines/planes.

a) Let $C = \{x^2 - y^2 = 3\}$ be a curve in $\mathbb{R}^2$ and $P = (2,1)$. Find a normal vector to the curve S at the point P and find both a parameterization of and an equation for the tangent line to S at P.

b) Let $S = \{z^2 - 2xyz = x^2 + y^2\}$ be a surface in $\mathbb{R}^3$ and $P = (1,2,-1)$. Find a normal vector to the surface S at the point P and find an equation of the tangent plane to S at P. Describe all other points of S whose tangent plane is parallel to the tangent plane your found above.

4. Let $f(x,y) = \frac{x-y}{x^2+1}$ and $P = (1,1,0)$. Find a normal vector to the graph of f at P. Find an equation of the tangent plane of the graph of f at P. Find a tangent vector to the graph of f at P pointing in the direction of steepest ascent.

Section II

5. Determine whether the following statements are always true, sometimes true/sometimes false, or always false. You do not need to justify your answer.

a) The curve $\gamma(t) = (3t+2, -2t)$ for $0 \le t \le 5$ passes through the origin.
b) The parameterization $\gamma(t) = (-\sin(t), -\cos(t))$ for $0 \le t \le 2\pi$ is for a unit circle going counterclockwise.
c) If a parameterized curve γ has constant speed (i.e. $\|\gamma'(t)\|$ is constant) then γ is a straight line.
d) Let $P \in \mathbb{R}^2$ and define a vector field by $\vec{F}(X) = \overrightarrow{PX}$ for $X = (x,y) \in \mathbb{R}^2$. The line integral of $\vec{F}$ around a closed curve γ depends on whether P is contained inside the region bounded by γ or not.
e) If a vector field $\vec{F}$ has constant scalar curl, then the line integral around *any* two circles of radius 1 is the same.

6. Compute the following line integrals $\int_\gamma \vec{F}$.

a) $\vec{F}(x,y) = 6x\,\vec{\imath} + (x+y^2)\,\vec{\jmath}$ and γ is the the interval on the y-axis from $(0,3)$ to $(0,5)$.
b) $\vec{F}(x,y) = (x^2+y)\,\vec{\imath} + (y^2+x)\,\vec{\jmath}$ and γ is along the parabola $y = x^2+1$ from $(0,1)$ to $(1,2)$.
c) $\vec{F}(x,y) = (\sin(x) + 2xy)\,\vec{\imath} + (\cos(y) + x^2)\,\vec{\jmath}$ and γ is the square with vertices $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$ going counterclockwise.
d) $\vec{F}(x,y) = (\sin(x) + xy)\,\vec{\imath} + (\cos(y) + x^2)\,\vec{\jmath}$ and γ is the square with vertices $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$ going counterclockwise. (Using Green's theorem will help here.)

7. For each vector field $\vec{F}$ find (and sketch) the region of definition, determine if the region is simply connected, and determine if $\vec{F}$ is path-independent.

a) $\vec{F}(x,y) = \frac{1}{y}\,\vec{\imath} - \frac{x+1}{y^2}\,\vec{\jmath}$
b) $\vec{F}(x,y) = \frac{x}{\sqrt{x^2+y^2-1}}\,\vec{\imath} + \frac{y}{\sqrt{x^2+y^2-1}}\,\vec{\jmath}$
c) $\vec{F}(x,y) = \frac{y}{\sqrt{x^2-1}}\,\vec{\imath} + \frac{1}{\sqrt{y^2-1}}\,\vec{\jmath}$
d) $\vec{F}(x,y) = x^2\sqrt{1-y-x^2}\,\vec{\imath} + 3\,\vec{\jmath}$

8. Change the order of integration of $\int_0^1 \int_0^{z^2} \int_y^2 f(x,y,z)\,dxdydz$ in all the other 5 possible ways.

9. Computing volumes of general solids.

a) Let a, b, c be positive numbers. Find the volume of the region between the coordinate planes and the plane $ax + by + cz = 1$.
b) Find the volume of the solid formed by drilling out a cylindrical hole of radius a through the center of a sphere of radius r. Of course, we assume that $0 \le a \le r$.

Section III

10. Determine whether the following statements are always true, sometimes true/sometimes false, or always false. You do not need to justify your answer.

a) If f and g are functions on the unit square $R = \{0 \le x \le 1,\ 0 \le y \le 1\} \subset \mathbb{R}^2$, then $\int_R f \cdot g = \int_0^1 f \cdot \int_0^1 g$.

b) The iterated integrals $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x,y,z)\,dzdydx$ and $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} f(x,y,z)\,dxdydz$ are equal.
c) If $\int_0^1 \int_0^2 f(x,y)\,dxdy = 1$ then $\int_0^2 \int_0^2 f(x,2y)\,dxdy = 2$.
d) A double integral can calculate the volume of a three dimensional solid region.
e) The flux of the constant vector field $\vec{\boldsymbol{k}}$ through a cube (oriented outward) sitting on the x-y-plane is always zero.
f) The flux of the constant vector field $\vec{\boldsymbol{k}}$ through a sphere with center at the origin is always zero.

11. Let R_a be the upper hemisphere of the solid sphere of radius $a > 0$ (some positive constant) centered at the origin. Calculate the integral $\int_{R_a} z\,dxdydz$. Hint: use spherical coordinates. (A side note for physics people: multiplied by $\frac{3}{2\pi a^3}$ (the reciprocal of the volume of R_a) this is the z-component of the *center of mass* of the solid R_a with constant density. By symmetry, you know that the x- and y-components are 0.)

12. Let R be the polyhedral region with vertices $(0,0)$, $(2,5)$, $(1,2)$, and $(3,7)$. Find a change of variables $x = x(s,t)$ and $y = y(s,t)$ so that this polyhedral region becomes a square. Calculate $\int_R xy^2\,dxdy$.

13. Let S be the region contained in the closed curve $x^2 - xy + y^2 = 1$. It's a tilted ellipse. Compute $\int_S xy\,dxdy$ by using the change of coordinates $x = s - \frac{1}{\sqrt{3}}t$, $y = s + \frac{1}{\sqrt{3}}t$.

14. Let S be the cyclinder $x^2 + y^2 = 1$ above the x-y-plane of height 4.
a) Find a parameterization for S, including limits.
b) Calculate the flux integral of $\vec{F}(x,y,z) = x\vec{\boldsymbol{\imath}} + y\vec{\boldsymbol{\jmath}}$ along S.

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