

Problem Set # 8 (due Wednesday 7 April 2010)

Material: Let $a \leq b$ be real numbers, and let $l(x)$ and $u(x)$ be continuous real functions satisfying $l(x) \leq u(x)$ for all $x \in [a, b]$. Consider $[a, b] \subset \mathbb{R}^2$ as an interval on the x -axis and the region $R \subset \mathbb{R}^2$ “above the interval $[a, b]$ and between the graphs of $l(x)$ and $u(x)$ ” defined by

$$R = \{ (x, y) \in \mathbb{R}^2 : a \leq x \leq b, l(x) \leq y \leq u(x) \}.$$

For each such region $R \subset \mathbb{R}^2$ and each continuous function $f : R \rightarrow \mathbb{R}$, we introduced a multivariable integral of f over R , which is equal to the iterated integral

$$\int_R f = \int_{x=a}^b \int_{y=l(x)}^{u(x)} f(x, y) dy dx.$$

It satisfies the following property: if $f(x, y) \geq 0$, then $\int_R f$ is the volume of the region above R (thought of in the x - y -plane) and below the graph of f .

Another way of computing volumes is using three-dimensional integrals. Let $a \leq b$ be real numbers, $l_y(x)$ and $u_y(x)$ be continuous single-variable real valued functions satisfying $l_y(x) \leq u_y(x)$ for all $x \in [a, b]$, and $l_z(x, y)$ and $u_z(x, y)$ be continuous two-variable real valued functions satisfying $l_z(x, y) \leq u_z(x, y)$ for all points (x, y) in the region $\{ (x, y) : a \leq x \leq b, l_y(x) \leq y \leq u_y(x) \}$ of the x - y -plane. Finally let

$$R = \{ (x, y, z) \in \mathbb{R}^3 : a \leq x \leq b, l_y(x) \leq y \leq u_y(x), l_z(x, y) \leq z \leq u_z(x, y) \}$$

and $f : R \rightarrow \mathbb{R}$ be a continuous function on the region R . Then there's a multivariable integral of f over R , which is equal to the iterated integral

$$\int_R f = \int_{x=a}^b \int_{y=l_y(x)}^{u_y(x)} \int_{z=l_z(x,y)}^{u_z(x,y)} f(x, y, z) dz dy dx.$$

It satisfies the following property: thinking of f as the “density” function of the solid region R , then $\int_R f$ is the mass of R , in particular, $\int_R 1$ is the volume of R .

Reading: CM 16.1-3.

1. CM 16.2 Exercise 20

Problems 32, 34, 36, 38, 42, 44, 48, 50

2. CM 16.3 Exercises 2 (note: a, b, c are arbitrary real constants), 6, 8, 10

Problems 16, 18