Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Spring 2010

Problem Set # 7 (due Friday 2 April 2010)

Material: Let $a \leq b$ be real numbers, and let $l(x)$ and $u(x)$ be continuous real functions satisfying $l(x) \leq u(x)$ for all $x \in [a, b]$. Consider $[a, b] \subset \mathbb{R}^2$ as an interval on the x-axis and the region $R \subset \mathbb{R}^2$ "above the interval $[a, b]$ and between the graphs of $l(x)$ and $u(x)$" defined by

$$R = \{ (x, y) \in \mathbb{R}^2 \,:\, a \leq x \leq b,\, l(x) \leq y \leq u(x) \}.$$

For example, if $l(x) = c$ and $u(x) = d$ are constant functions, then R is the box with corners (a, c), (b, c), (a, d), and (b, d). If $a = 0$, $b = 1$, $l(x) = 0$ and $u(x) = x$, then R is a triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$. If $a = -1$, $b = 1$, $l(x) = -\sqrt{1 - x^2}$, and $u(x) = \sqrt{1 - x^2}$, then R is a circle of radius 1 centered at the origin.

For each region $R \subset \mathbb{R}^2$ and each continuous function $f : R \to \mathbb{R}$, we introduced a multivariable integral of f over R

$$\int_R f$$

satisfying the property that if $f(x, y) \geq 0$, then $\int_R f$ is the area above R (thought of in the x-y-plane) and below the graph of f.

If the region $R \subset \mathbb{R}^2$ is defined by a, b, $l(x)$, and $u(x)$ as above, then we write the multivariable integral of f over R as

$$\int_R f = \int_{x=a}^{b} \int_{y=l(x)}^{u(x)} f(x, y)\, dy\, dx$$

and we can perform "iterated integration" just as in Fubini's theorem: first find an antiderivative of $f(x, y)$ with respect to y, i.e. a function $g(x, y)$ so that $\frac{\partial g}{\partial y} = f(x, y)$, then

$$\begin{aligned} \int_{x=a}^{b} \int_{y=l(x)}^{u(x)} f(x, y)\, dy\, dx &= \int_{x=a}^{b} \left(g(x, y) \Big|_{y=l(x)}^{u(x)} \right) dx \\ &= \int_{x=a}^{b} \big(g(x, u(x)) - g(x, l(x)) \big)\, dx \end{aligned}$$

and then integrate with respect to x in the usual way.

We can also define regions $R \subset \mathbb{R}^2$ "in the other direction" by

$$R = \{ (x, y) \in \mathbb{R}^2 \,:\, c \leq y \leq d,\, p(y) \leq x \leq q(y) \},$$

where $c \leq d$ and $p(y) \leq q(y)$ for all $y \in [c, d]$. Then we write the multivariable integral of f over R as

$$\int_R f = \int_{y=c}^{d} \int_{x=p(y)}^{q(y)} f(x, y)\, dx\, dy$$

and we can perform "iterated integration" in the same way, by first finding an antiderivative of $f(x, y)$ with respect to x. You can find worked out examples of this in CM 16.2. This homework assignment is a mild introduction to this concept.

Reading: CM 16.1-2.

1. CM 16.1 Problem 26

2. CM 16.2 Exercises 2, 4, 6, 8, 10, 12, 14, 16, 18