Emory University Department of Mathematics & CS
Math 211 Multivariable Calculus
Fall 2011

Problem Set # 8 (due Friday 28 October 2011)

Material: Let $a \leq b$ be real numbers, and let $l(x)$ and $u(x)$ be continuous real functions satisfying $l(x) \leq u(x)$ for all $x \in [a, b]$. Consider $[a, b] \subset \mathbb{R}^2$ as an interval on the x-axis and the region R of $\mathbb{R}^2$ "above the interval $[a, b]$ and between the graphs of $l(x)$ and $u(x)$" defined by

$$R = \{ (x, y) \in \mathbb{R}^2 \, : \, a \leq x \leq b, \, l(x) \leq y \leq u(x) \}.$$

For example, if $l(x) = c$ and $u(x) = d$ are constant functions, then R is the box with corners (a, c), (b, c), (a, d), and (b, d). If $a = 0$, $b = 1$, $l(x) = 0$ and $u(x) = x$, then R is a triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$. If $a = -1$, $b = 1$, $l(x) = -\sqrt{1 - x^2}$, and $u(x) = \sqrt{1 - x^2}$, then R is a circle of radius 1 centered at the origin.

For each region R in $\mathbb{R}^2$ and each continuous function $f : R \to \mathbb{R}$, we introduced a multivariable integral of f over R, which is equal to the iterated integral

$$\int_R f = \int_{x=a}^{b} \int_{y=l(x)}^{u(x)} f(x, y) \, dy \, dx.$$

and satisfies the following property: if $f(x, y) \geq 0$, then $\int_R f$ is the volume of the region above R (thought of in the x-y-plane) and below the graph of f.

Another way of computing volumes is using three-dimensional integrals. Let $a \leq b$ be real numbers, $l_y(x)$ and $u_y(x)$ be continuous single-variable real valued functions satisfying $l_y(x) \leq u_y(x)$ for all x in $[a, b]$, and $l_z(x, y)$ and $u_z(x, y)$ be continuous two-variable real valued functions satisfying $l_z(x, y) \leq l_z(x, y)$ for all points (x, y) in the region $\{ (x, y) \, : \, a \leq x \leq b, \, l_y(x) \leq y \leq u_u(x) \}$ of the x-y-plane. Finally let

$$R = \{ (x, y, z) \in \mathbb{R}^3 \, : \, a \leq x \leq b, \, l_y(x) \leq y \leq u_y(x), \, l_z(x, y) \leq z \leq u_z(x, y) \}$$

and $f : R \to \mathbb{R}$ be a continous function on the region R. Then there's a multivariable integral of f over R, which is equal to the iterated integral

$$\int_R f = \int_{x=a}^{b} \int_{y=l_y(x)}^{u_y(x)} \int_{z=l_z(x,y)}^{u_z(x,y)} f(x, y, z) \, dz \, dy \, dx.$$

satisfying the following property: thinking of f as the "density" function of the solid region R, then $\int_R f$ is the mass of R, in particular, $\int_R 1$ is the volume of R.

Reading: CM 16.1-3.

1. CM 16.2 Exercise 20
Problems 32, 34, 35, 37, 36, 38, 42, 44, 48, 50

2. CM 16.3 Exercises 2 (note: a, b, c are arbitrary real constants; your answer should be expressed in terms of them!), 6, 8, 10
Problems 16 (note that above the first quadrant ($x \geq 0$ and $y \geq 0$) of the x-y-plane, the plane $3x + 4y + z = 6$ is below the plane $2x + 2y + z = 6$ and you are interested in finding the volume of the region *between* the two planes over the given triangle $x + y \leq 1$ in the x-y-plane),
Problems 18 ("Under the sphere" really means "Inside the sphere").

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