MATH 170 IDEAS IN MATHEMATICS (SUMMER 2006)
Problem Set 9: Extra Credit

Due by 4 pm, Friday, June 23rd

Note: This problem set is optional. You may hand in any or all of the problems to receive extra homework credit. Please don't ask me exactly which problem is worth how much extra credit. Trust me, I'll be very fair in figuring that out once you've turned it in. On average though, I'd say each of the following problems is worth 10 homework points.

1. Numbers from MIU

Definition 1. Recall the definition of *reduced* theorems of the MIU system, namely, that a theorem of the MIU-system is reduced if no shorter string can be derived (using any number of steps, including lengthening and shortening rules) starting from it. It's not clear exactly which theorems of MIU are reduced, but you should have showed in Problem Set 1 that at least theorems of the form

$$\mathsf{M}i_0\mathsf{U}i_1\mathsf{U}i_2\mathsf{U}\dots i_n\mathsf{U},$$

are reduced, where $i_0, i_1, \dots$ above stand for either of I or II. Call strings of that very particular shape *nice*. So we know all nice theorems are reduced. Examples of nice theorems are

MIU, MIIU, MIUIU, MIIUIIU, MIUIIUIU, and MIIUIUIIUIIUIUIIU.

Now to each nice string, we can associate a number, which we'll call its *niceness*, as follows. The general nice string

$$\mathsf{M}i_0\mathsf{U}i_1\mathsf{U}i_2\mathsf{U}\dots i_n\mathsf{U},$$

has niceness number equal to

$$j_0 + 2 \cdot j_1 + 4 \cdot j_2 + 8 \cdot j_3 + \cdots + 2^n \cdot j_n,$$

where for $k = 0, 1, \dots, n$,

$$j_k = \begin{cases} 0 & \text{if } i_k \text{ is } \mathsf{I} \\ 1 & \text{if } i_k \text{ is } \mathsf{II} \end{cases}.$$

Note that the string of digits $j_0 j_1 j_2 \dots j_n$ is the binary expansion of the associated niceness number.

For example

nice theorem	$j_0 j_1 \dots j_n$	niceness number		
MIU	0	0	$=$	0
MIIU	1	1	$=$	1
MIUIU	00	$0 + 2 \cdot 0$	$=$	0
MIIUIIU	11	$1 + 2 \cdot 1$	$=$	3
MIUIIUIU	010	$0 + 2 \cdot 1 + 4 \cdot 0$	$=$	2
MIIUIUIIUIIUIUIIU	101101	$1 + 2 \cdot 0 + 4 \cdot 1 + 8 \cdot 1 + 16 \cdot 0 + 32 \cdot 1$	$=$	45

All the strings in the above table are actually theorems. So $0, 1, 2, 3$ and 45 are each the niceness number of some nice theorem of the MIU-system. We'll call such numbers *nice numbers*. Now answer the following questions:

i. Is 7 a nice number, i.e. is there a nice theorem with niceness 7? Is 4 a nice number? Be careful. For example, even though the string MIUIIU (i.e. the smallest string with niceness 2) is not a theorem of the MIU-system (why is that?) the string MIUIIUIU is in fact a theorem and has niceness 2, so sometimes the smallest string with a particular niceness isn't a theorem, but some larger one is!

ii. Note that given a nice string, applying rule 2 of the MIU-system gives you back a nice string. What does rule 2 do to the niceness numbers of nice strings in terms of the string and the number? Can you write down a formula?

iii. List the nice numbers that are ≤ 15.

iv. Is the sequence of nice numbers, i.e. numbers that occur as the niceness of some nice theorem, a recursively enumerable set of natural numbers? Do you think it's a recursive set of natural numbers?

2. XOR as symmetry.

a. Recall the logical operation of *exclusive or* or XOR given by the truth table:

P	Q	P XOR Q
T	T	F
T	F	T
F	T	T
F	F	F

In the following, let 0 stand for F and 1 stand for T. Then we can write the action of XOR as a "multiplication"

$$1 \texttt{ XOR } 1 = 0, \quad 1 \texttt{ XOR } 0 = 1, \quad 0 \texttt{ XOR } 1 = 1, \quad 0 \texttt{ XOR } 0 = 0,$$

and we can extend this "multiplication" digit by digit to two digit numbers by writing

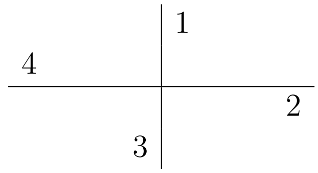
$$p_1p_2 \texttt{ XOR } q_1q_1 = (p_1 \texttt{ XOR } q_1)(p_2 \texttt{ XOR } q_2),$$

i.e. first apply XOR to the first digits of the two numbers, then to the second digits. For example

$$00 \texttt{ XOR } 10 = 10 \quad \text{and} \quad 01 \texttt{ XOR } 11 = 10.$$

Now write out the "multiplication" table for XOR acting on the two digit strings $00, 10, 01, 11$. What do you notice about this table.

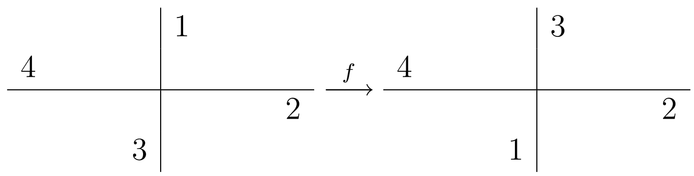
b. Consider the numbered cross of uneven arm lengths (or just cross from now on):

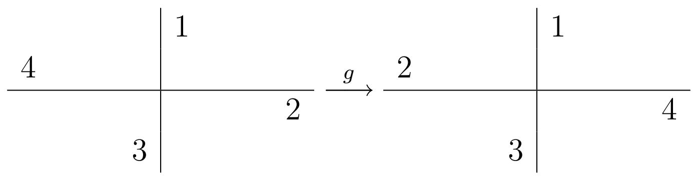


Answer the following questions:

i. Write down (pictorially) all the symmetries of the cross as you did for the square in Problem Set 2.

ii. Define the two “flips” about the major axis:





As always, denoting the “do nothing” symmetry by n, write your symmetries from part **a.** in terms of n, f, g using the same rules of composition (multiplication) of symmetries from Problem Set 2.

iii. Write out the multiplication table for the symmetries of the cross.

iv. Show there’s an isomorphism between the logical operation `XOR` acting on two digit strings and the symmetries of the cross.

3. More derivations in PC.

Find derivations for the following theorems of PC.

i. $<<P \Rightarrow \neg <Q \Rightarrow V>> \Rightarrow <P \Rightarrow Q>>$

ii. $<<P \Rightarrow <Q \wedge \neg R>> \Rightarrow <P \Rightarrow <R \Rightarrow Q>>>$

iii. $<<<P \Rightarrow R> \wedge <Q \Rightarrow R>> \Rightarrow <<P \vee Q> \Rightarrow R>>$

iv. $<<<P \wedge <Q \vee R>> \Rightarrow <Q \wedge R>> \Rightarrow <P \Rightarrow <Q \Rightarrow R>>>$