Dartmouth College Department of Mathematics
Math 125 Current Problems in Number Theory:
Galois Cohomology and Descent
Winter 2022

Group Work # 2 (Thursday, February 3)

Reading: *Gill–Szamuely* §1.1-1.2, 2.1-2.2, 4.1, *Serre* §1.1-1.4, *Shatz* Ch. I, IV.1

Group Work: To be discussed during the second half of class on Thursday, with the discussion led by a student selected ahead of time.

1. *Pontryagin duality* Let G be a locally compact (i.e., every point has a compact neighborhood) Hausdorff topological group. For example, $\mathbb{R}$ or any compact group or any discrete group is locally compact. Let $U \subset \mathbb{C}^\times$ be the unit circle, which is a locally compact topological group. Define the **Pontryagin dual** $\check{G}$ to be the group of all continuous homomorphisms $\phi : G \to U$ equipped with the compact-open topology (look this up if you don't know what it is).

(a) Prove that $\check{\mathbb{Z}} \cong U$ and that $\check{U} \cong \mathbb{Z}$.

(b) Prove that $\check{\mathbb{R}} \cong \mathbb{R}$ via a map (in the other direction) $x \mapsto (y \mapsto e^{2ixy})$.

(c) Prove that if G is a finite abelian group, then $\check{G} \cong G$.

(d) Prove that if G is a discrete torsion group, then $\check{G}$ is a profinite group.

(e) Prove that $\widecheck{\mathbb{Q}/\mathbb{Z}} \cong \hat{\mathbb{Z}}$. **Hint.** $\mathbb{Q}/\mathbb{Z}$ is a direct limit of cyclic groups.

2. Let A be an (associative unital) F-algebra. We say that an F-linear map $^- : A \to A$ is an **involution** if $\overline{1} = 1$, $\overline{\overline{a}} = a$ for all $a \in A$, and $\overline{ab} = \overline{b}\overline{a}$ for all $a, b \in A$. An involution is called **standard** if $a\overline{a} \in F$ for all $a \in A$. As usual, we consider $F \subset A$ as the F-subspace spanned by the identity in A.

(a) Prove that if $^-$ is a standard involution on an F-algebra A then $a + \overline{a} \in F$ for all $a \in A$. **Hint.** Consider $(1 + a)(\overline{1 + a})$.

(b) If $^-$ is a standard involution on an F-algebra A, define the **(involution) trace** $\mathrm{T} : A \to F$ by $a \mapsto a + \overline{a}$ and the **(involution) norm** $\mathrm{N} : A \to F$ by $a \mapsto a\overline{a}$. Prove that any $a \in A$ satisfies $a^2 - \mathrm{T}(a)a + \mathrm{N}(a) = 0$. This is an analogue of the Cayley–Hamilton theorem and one often calls $x^2 - \mathrm{T}(a)x + \mathrm{N}(a) \in F[x]$ the involution characteristic polynomial of $a \in A$.

(c) Prove that if K is an F-algebra of dimension 2, then K is commutative and admits a unique standard involution. What is this in the case that K/F is a separable extension of degree 2? What about $K = F \times F$? What about the dual numbers $K = F[x]/(x^2)$?

(d) Prove that if A is a quaternion algebra over F, then A has a unique standard involution. **Hint.** Restrict to a quadratic extension contained in A.

3. About division algebras.

(a) Over an algebraically closed field F, the only finite dimensional division F-algebra if F itself. **Hint.** Use the existence of eigenvalues of linear operators on finite dimensional vector spaces over algebraically closed fields as indicated in class.

(b) Let $A = \mathbb{C}(t)$ the rational function field over the complex numbers. Then A is an infinite dimensional division $\mathbb{C}$-algebra (but clearly far from central). Where does your previous argument break down for A? (Fun fact: it turns out that there are no nontrivial central division $\mathbb{C}(t)$-algebras, this is a consequence of Tsen's Theorem. Can you find a new proof of this fact?)

(c) Prove that if A is a (nonsplit) quaternion algebra over a field F (of characteristic not 2) and K/F is a quadratic extension with $K \subset A$ a sub F-algebra, then $A \otimes_F K$ is split.

(d) Read the proof of *Gille–Szamuely* Lemma 2.4.4, really Theorem 2.2.1, really Lemma 2.2.2. This was not as easy as I made it appear in class!

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