Dartmouth College Department of Mathematics
Math 125 Current Problems in Number Theory:
Galois Cohomology and Descent
Winter 2022

Group Work # 1 (Thursday, January 13th)

Reading: *Milne* Ch. 7, *Gill–Szamuely* §4.1, *Serre* §1.1-1.4, *Shatz* Ch. I, IV.1

Group Work: To be discussed during the second half of class on Thursday, with the discussion led by a student selected ahead of time.

1. Understand the following facts about subgroups H of a topological group G.

(a) The closure $\overline{H}$ of any subgroup H of G is a subgroup.

(b) The closure $\overline{H}$ of any normal subgroup H of G is a normal subgroup.

(c) The coset space G/H, with the quotient topology, is a topological group if H is a normal subgroup of G.

(d) The trivial subgroup $\{1_G\}$ is closed if and only if G is Hausdorff. Thus $G/\overline{\{1_G\}}$ is a Hausdorff topological group (called the **Kolmogorov quotient**).

(e) The coset space G/H, with the quotient topology, is Hausdorff if and only if H is closed in G. (This is why one generally sticks to closed subgroups when taking quotients of topological groups.)

(f) The connected component of the identity G° in G is a closed normal subgroup. The quotient G/G° is a totally disconnected Hausdorff topological group.

(g) If G is a connected topological group, then G does not contain any proper open subgroups.

(h) Every open subgroup H of G is closed.

(i) Every finite index closed subgroup H of G is open.

(j) If G is compact then every open subgroup H has finite index in G.

2. For an abstract group G, define the **profinite topology** on G to be the one with a basis of open subsets consisting of all (left) cosets of all subgroups of finite index.

(a) Why can you take the basis to consist of all cosets (left and right) of all normal subgroups of finite index and get the same topology?

(b) What is the profinite topology on the additive groups $\mathbb{Z}$ and $\mathbb{R}$?

3. The **profinite completion** $\widehat{G}$ of an abstract group G is defined to be the inverse limit $\varprojlim_N G/N$ over the inverse system of quotients G/N indexed by the normal subgroups N of finite index in G, ordered so that for any inclusion $N \subset N'$ we consider the induced quotient $G/N' \to G/N$.

(a) Prove that there is a canonical homomorphism $G \to \widehat{G}$, which is universal for homomorphisms from G to profinite groups.

(b) What is the profinite completion of the additive groups $\mathbb{Z}$ and $\mathbb{R}$?

4. Prove that the following statements are equivalent for an abstract group G.

(a) The profinite topology on G is Hausdorff.

(b) The intersection $\bigcap U$, taken over all finite index normal subgroups U of G, is the trivial subgroup $\{1_G\}$.

(c) The canonical homomorphism $G \to \widehat{G}$ is injective.

A group G satisfying these conditions is called **residually finite**. Any profinite group is residually finite. Prove that any free group is residually finite.

Residually finite groups are important in geometric group theory, as they contain the class of fundamental groups of compact 3-manifolds.

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