NAME : ANSWERS

SECTION : (circle one) Weber (10 Hour) Mainkar (12 hour)

Math 8

11 March 2008

Final Exam

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have two hours and you should attempt all problems.

- **Except in Problem 13, you must show all work and give a reason (or reasons) for your answer. A CORRECT ANSWER WITH INCORRECT WORK WILL BE CONSIDERED WRONG.**
- ***Print*** your name in the space provided and circle your instructor's name.
- Calculators or other computing devices are not allowed.
- Use the blank page at the end of the exam for scratch work.

1. (10) Evaluate $\int x \ln x \, dx$.

Integration by parts.

$u = \ln x \qquad v = x^2/2$

$du = \frac{1}{x}\,dx \qquad dv = x\,dx$

$$\int x \ln x \, dx = \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \frac{1}{x}\,dx$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

2. (12) Evaluate $\int_0^{\frac{1}{\sqrt{2}}} \frac{x^2}{\sqrt{1-x^2}}\,dx$.

Trig substitution.

$x = \sin\theta$ $\qquad$ $\sin(\pi/4) = \frac{1}{\sqrt{2}}$

$dx = \cos\theta\, d\theta$ $\qquad$ $\sin(0) = 0$

$$\int_0^{1/\sqrt{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

$$= \int_0^{\pi/4} \sin^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{\theta}{2} - \frac{\sin 2\theta}{4}\Big|_0^{\pi/4}$$

$$= \pi/8 - \frac{1}{4}.$$

3. (10) Determine whether the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\cos(n^4) + \sin(n^5)}{n^9}$$

Mention any test(s) that you might use and verify that it is applicable.

Comparison Test:

$$\sum_{n=1}^{\infty} \left| \frac{\cos(n^4) + \sin(n^5)}{n^9} \right| \leq \underbrace{\sum_{n=1}^{\infty} \frac{2}{n^9}}_{\text{convergent p-series.}}$$

So the original series is absolutely convergent.

4. (12) Find the radius of convergence and the interval of convergence of the following series.

$$\sum_{n=1}^{\infty} \frac{(-5)^{n+2}(x-1)^n}{n^2}$$

Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-5)^{n+2}(x-1)^n}{n^2}\right|} = |5x-5|.$$

$$|5x-5| < 1 \quad \text{for} \quad \frac{4}{5} < x < \frac{6}{5}.$$

Check endpoints:

$$x = \frac{4}{5}: \sum_{n=1}^{\infty} \frac{(-5)^{n+2}\left(-\frac{1}{5}\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Convergent p-series

$$x = \frac{6}{5}: \sum_{n=1}^{\infty} \frac{(-5)^{n+2}\left(\frac{1}{5}\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

convergent p-series

Interval of convergence: $\frac{4}{5} \leq x \leq \frac{6}{5}$.

Radius of convergence: $1/5$

5. (14) Find the first 2 nonzero terms in the Maclaurin series for $f(x) = \tan x$.

$f(x) = \tan x$	$f(0) = 0$
$f'(x) = \sec^2 x$	$f'(0) = 1$
$f''(x) = 2\sec^2 x \tan x$	$f''(0) = 0$
$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$	$f'''(0) = 2$

First 2 nonzero terms:

$$x + \frac{2x^3}{3!} = x + \frac{x^3}{3}.$$

6. (12) Find an equation of the plane which contains the x-axis as well as the line given by the parametric equations $x = t$, $y = 2t$, $z = 3t$.

Line #1 : $\langle 1, 0, 0 \rangle s$ (x-axis)

Line #2 : $\langle 1, 2, 3 \rangle t$

Perpendicular to both:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= -3\vec{j} + 2\vec{k}$$

Point on plane: $(0,0,0)$

Plane: $0(x-0) + -3(y-0) + 2(z-0) = 0.$

7. (10) Find the arc length of the curve $s(t) = \langle e^t \sin t, e^t \cos t, 1 \rangle$ from $t = 0$ to $t = 1$.

Arc length $= \int$ speed

Speed $= |\vec{v}|$

$$\vec{v}(t) = (e^t \sin t + e^t \cos t)\vec{i} + (e^t \cos t - e^t \sin t)\vec{j}$$

$$\text{Speed} = \sqrt{e^{2t}(2\sin^2 t + 2\cos^2 t)}$$

$$= e^t \sqrt{2}$$

$$\text{Arc length} = \sqrt{2}\int_0^1 e^t \, dt = \sqrt{2}\, e^t \Big|_0^1 = \sqrt{2}\, e - \sqrt{2}$$

8. (10) The gradient of $f(x, y, z)$ is

$$\nabla f = \langle 2xyz + 2e^z, x^2z - \cos y, x^2y + 2e^z \rangle,$$

where $x = s^2t$, $y = t^3$, and $z = e^s$. What is $\frac{\partial f}{\partial s}$? You need not simplify your answer (but it should contain no ∂ symbols).

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}.$$

$$= (2xyz + 2e^z)(2st)$$
$$+ (x^2z - \cos y)(0)$$
$$+ (x^2y + 2e^z)(e^s).$$

9. (12) Find an equation of the line through $(1, 2, 0)$ which is orthogonal to the tangent plane of the surface given by $\underbrace{z\cos(xy) + x^2 - y^3 z}_{F(x,y,z)} = 1$ at the point $(\pi, 1, \pi^2 - 1)$.

Tangent plane to a level surface.

$$\nabla F = \langle -yz \sin(xy) + 2x, -xz \sin(xy) - 3y^2 z, \cos(xy) - y^3 \rangle$$

$$\nabla F(\pi, 1, \pi^2 - 1) = \langle 2\pi, -3\pi^2 + 3, -2 \rangle.$$

Plane:

$$2\pi(x - \pi) + (-3\pi^2 + 3)(y - 1) - 2(z - \pi^2 + 1) = 0.$$

Line:

$$\vec{r}(t) = \langle 2\pi, -3\pi^2 + 3, -2 \rangle t + \langle 1, 2, 0 \rangle$$

10. (14) Consider the function $f(x, y) = x^3 + y^2 - xy$. At the point $(1, 1)$, in what direction(s) is the rate of change of f equal to zero? Please give your answer as one or more *unit* vectors.

$$\nabla f = \langle 3x^2 - y,\ 2y - x \rangle$$

$$\nabla f(1,1) = \langle 2, 1 \rangle.$$

Let $\vec{u} = \langle a, b \rangle$. If $D_{\vec{u}} f = 0$ then:

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = 2a + b = 0,$$

So $b = -2a$.

$$\frac{\vec{u}}{|\vec{u}|} = \frac{\langle a, -2a \rangle}{\sqrt{5a^2}} = \frac{\langle 1, -2 \rangle}{\sqrt{5}} \text{ and } \frac{\langle -1, 2 \rangle}{\sqrt{5}}$$

11. (14) (a) Find the rate of change of the function $f(x, y) = \sqrt{24 - x^2 - y^2}$ at the point $(4, -2)$ in the direction given by $\theta = \frac{\pi}{6}$.

$$\vec{u} = \langle \cos \pi/6, \sin \pi/6 \rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$$

$$\nabla f = \left\langle \frac{-x}{\sqrt{24 - x^2 - y^2}}, \frac{-y}{\sqrt{24 - x^2 - y^2}} \right\rangle$$

$$\nabla f(4, -2) = \langle -2, 1 \rangle$$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = -\sqrt{3} + \frac{1}{2}.$$

(b) In what direction does f of question 11(a) attain its maximum rate of change at the point $(4, -2)$? (You need not specify the direction by an angle.)

$$\frac{\nabla f}{|\nabla f|} = \frac{\langle -2, 1 \rangle}{\sqrt{5}} = \langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle.$$

12. (20) Find and classify all the critical points of the function $f(x, y) = 3x - x^3 - 3xy^2$.

$f_x = 3 - 3x^2 - 3y^2$

$f_y = -6xy$

If $f_y = 0$ then $x = 0$ or $y = 0$.

If $x = 0$ then $f_x = 3 - 3y^2$, so need $y = \pm 1$.

If $y = 0$ then $f_x = 3 - 3x^2$, so need $x = \pm 1$.

Critical points: $(0, -1)$, $(0, 1)$, $(-1, 0)$, $(1, 0)$.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -6x & -6y \\ -6y & -6x \end{vmatrix} = 36x^2 - 36y^2.$$

$D(0, -1) = -36$ saddle point.

$D(0, 1) = -36$ saddle point.

$D(-1, 0) = 36$, $f_{xx}(-1, 0) = 6$ Local min.

$D(1, 0) = 36$, $f_{xx}(1, 0) = -6$ local max.

13. (25) For each of the following statements, fill in the blank with the letters **T** of **F** depending on whether the statement is true or false. You do not need to show your work and no partial credit will be given on this problem.

(a) If $\lim_{(x,y)\to(2,4)} f(x,y) = 3$, then f is continuous at $(2,4)$.

False – What if $f(2,4) \neq 3$?

ANS: F

(b) Suppose the linearization of a function $z = f(x,y)$ at the point $(4,2)$ is $L(x,y) = 3x + y - 3$. Then the differential as x and y change from $(4,2)$ to $(3,3)$ is $dz = 9$.

$\nabla f = \langle 3, 1 \rangle$ (read off from Linearization)

$D_{\frac{\langle -1, 1\rangle}{\sqrt{2}}} = \frac{-3}{\sqrt{2}} + \frac{1}{\sqrt{2}} = -\sqrt{2}$

ANS: F

(c) The sequence $\{(-1)^n(1 - \frac{1}{n})\}$ is convergent.

The terms approach 1 in absolute value, but alternate in sign.

ANS: F

(d) The series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$ is divergent.

The terms of the sequence do not tend to 0. Series diverges by Test for divergence.

ANS: T

(e) $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{a}) = 0.$

$\vec{b} \times \vec{a}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

ANS: T