Hour Exam #1

Math 3

Oct. 10, 2012

Name (Print): ____

Last First

On this, the first of the two Math 3 hour-long exams in Fall 2012, and on the second hour-exam, and on the final examination I will work individually, neither giving nor receiving help, guided by the Dartmouth Academic Honor Principle.

Signature:___________________________________

Instructor (circle):

Lahr (Sec. 1, 8:45) Diesel (Sec. 2, 10:00)

Dorais (Sec. 3, 11:15) Dorais (Sec. 4, 12:30)

Wolff (Sec. 5, 1:45)

Instructions: You are not allowed to use calculators, books, or notes of any kind. All of your answers must be marked on the Scantron form provided or entered on the test, depending on the problem. Take a moment now to print your name and section clearly on your Scantron form and on page 1 of your exam booklet and sign the affirmation. You may write on the exam, but you will only receive credit on Scantron (multiple-choice) problems for what you write on the Scantron form. At the end of the exam, you must turn in both your Scantron form and your exam booklet. There are 15 multiple-choice problems worth 4 points each and 3 long-answer written problems worth a total of 40 points. Check to see that you have 11 pages of questions plus the cover page for a total of 12 pages.

Non-multiple choice questions:

Problem	Points	Score
16	15	
17	10	
18	15	
Total	40	

For this page, let $f(x) = \dfrac{x+4}{x^2+x-12}$.

1. What is $\lim_{x\to-\infty} f(x)$?

   (a) 0
   (b) $-\infty$
   (c) -1
   (d) The limit does not exist
   (e) None of the above

Answer a)

$\dfrac{x+4}{x^2+x-12} = \dfrac{x+4}{(x+4)(x-3)}$ so $g(x) = \dfrac{1}{x-3}$ extends f to include the point $x=-4$. Moreover, $f(x) = g(x)$ where both functions are defined, $\mathbb{R}\setminus\{-4,3\}$. Thus $\lim_{x\to-\infty} \dfrac{x+4}{x^2+x-12} = \lim_{x\to-\infty} \dfrac{1}{x-3} = 0$

2. What is the domain of $f(x)$?

   (a) $(-\infty,-4)\cup(-4,\infty)$
   (b) $(-\infty,\infty)$
   (c) $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$
   (d) $(-\infty,-4)\cup(-4,6)\cup(6,\infty)$
   (e) None of the above

Answer c)

f is defined everywhere except at points where $x^2+x-12=0$

$x^2+x-12 = (x-3)(x+4)$, so $x^2+x-12=0$ only at $x=3$ and $x=-4$

$\therefore$ the domain is $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$.

For this page, let $f(x) = \dfrac{x^3 + 4x^2 + 4x}{(x+2)(x+3)}$.

3. What are the vertical asymptotes f?

   (a) $x = -3$

   (b) $x = 2$ and $x = 3$

   (c) $x = 2$ and $x = -3$

   (d) $y = \sqrt[3]{-8}$

   (e) None of the above

Answer a) f is defined everywhere except at $x=-2$ and $x=-3$.

$$\lim_{x\to -2} \frac{x^3+4x^2+4x}{(x+2)(x+3)} = \lim_{x\to -2} \frac{x(x+2)^2}{(x+2)(x+3)} = \lim_{x\to -2} \frac{x^2+2x}{x+3} = \frac{0}{1} = 0$$

so f doesn't have a vertical asymptote at $x=-2$.

On the other hand $\lim_{x\to -3^+} \frac{x^2+2x}{x+3} = \infty$ and $\lim_{x\to -3^-} \frac{x^2+2x}{x+3} = -\infty$

This is because $x^2+2x > 0$ as we approach from left or right, but $x+3 > 0$ as we approach from the right and $x+3 < 0$ as we approach from the left.

4. What are the horizontal asymptotes of f?

   (a) $y = -2$

   (b) $y = 4/3$

   (c) $y = 0$

   (d) $y = -2$ and $y = -3$

   (e) None of the above

Answer e)

As in the previous answer, $\lim_{x\to -3^+} f = \infty$, $\lim_{x\to -3^-} f = -\infty$

So there can't be a horizontal asymptote.

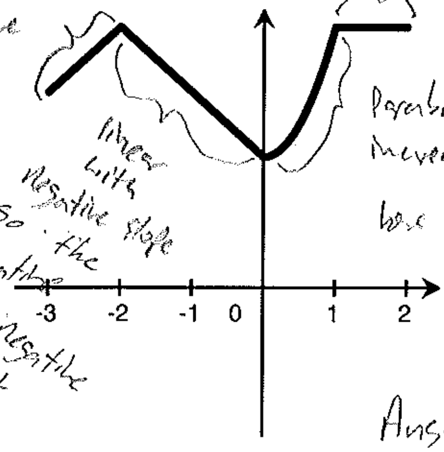
5. Suppose the graph of the function $f(x)$ shown below has three linear pieces and one parabolic (polynomial of degree 2) piece

linear with positive slope So the derivative is a positive constant

linear with negative slope So the derivative is a negative constant

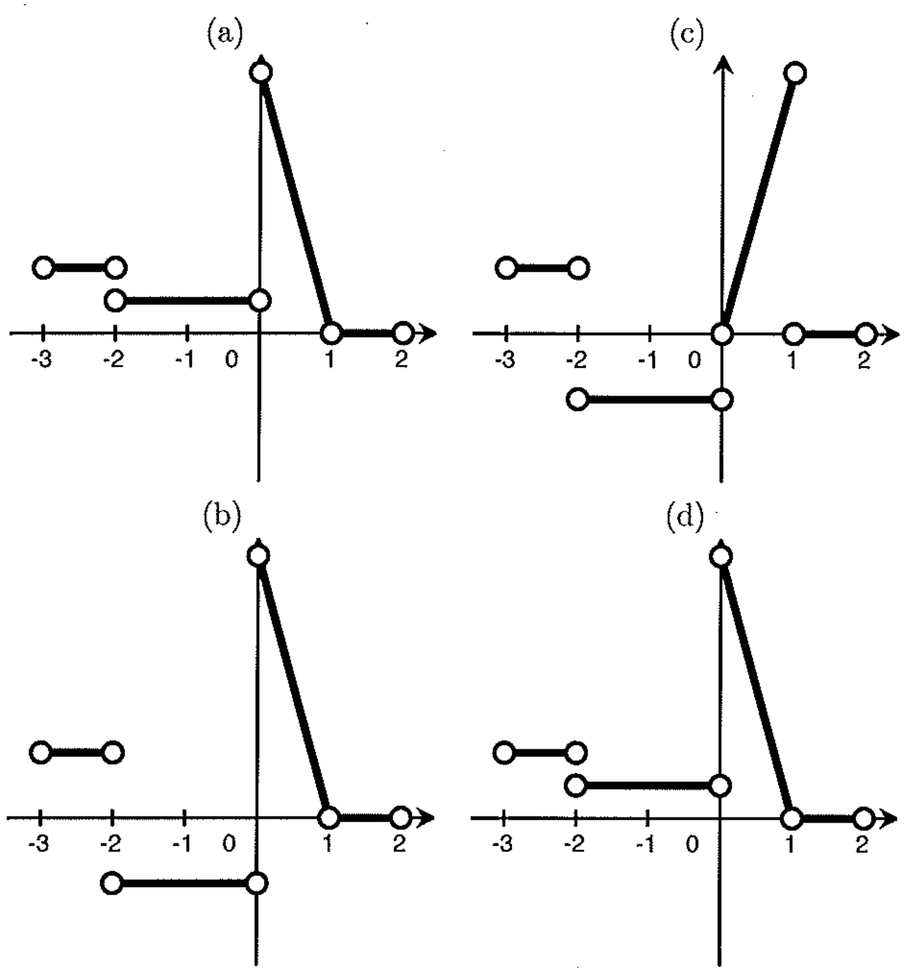
constant So the derivative is 0

Parabolic and increasing, So the derivative here is linear with positive values



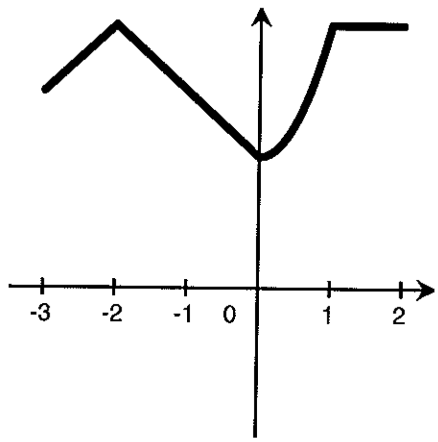
Answer c)

Of the following graphs, which could be the graph of $f'(x)$?



(e) None of the above

6. Again, suppose the graph of the function $f(x)$ looks like

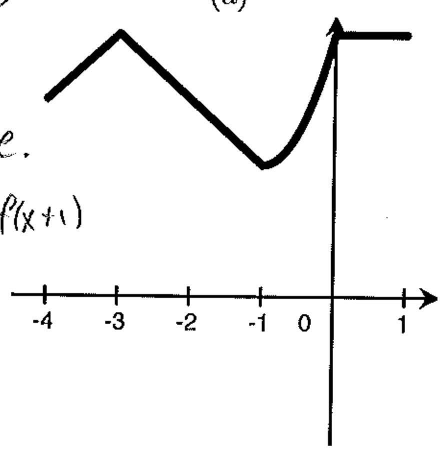


Answer c)

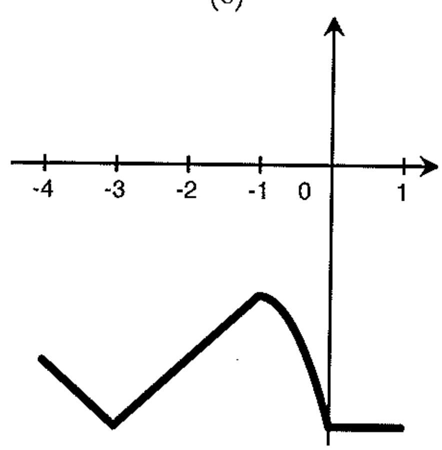
Of the following graphs, which is the graph of $-f(x+1)$?

Going from $f(x)$ to $f(x+1)$ we shift to the left by one.
The from $f(x+1)$ to $-f(x+1)$ we reflect about the x-axis.

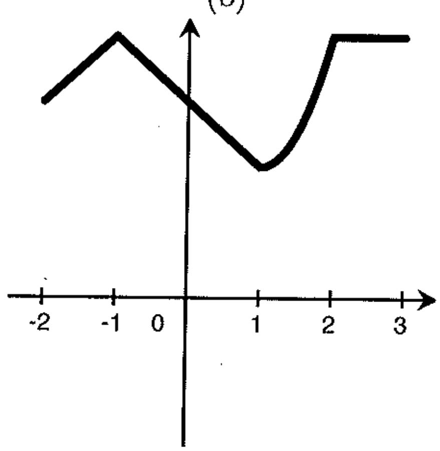
(a)



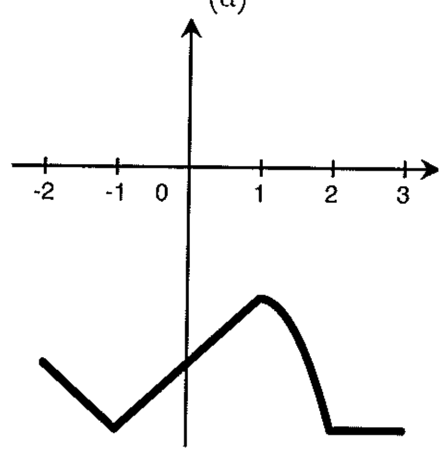
(c)



(b)



(d)



(e) None of the above

For this page, let $f(x) = \dfrac{1}{x}$ and $g(x) = \sqrt{3 - \sqrt{2-x}}$.

7. What is the domain of $f(g(x))$?

   (a) $x \le 2$

   (b) $-7 < x \le 2$

   (c) $2 \le x < 11$

   (d) $x \ne 0$

   (e) $x < 2$ or $x > 11$

Answer b) $f(g(x)) = \dfrac{1}{\sqrt{3-\sqrt{2-x}}}$, $\sqrt{3-\sqrt{2-x}}$ is not defined on $\mathbb{R}$ when $3 - \sqrt{2-x} < 0$, $\therefore$ $3 - \sqrt{2-x} \ge 0 \Rightarrow 3 \ge \sqrt{2-x}$ $\Rightarrow 9 \ge 2-x \Rightarrow x \ge -7$. If $x = -7$, then $\sqrt{3-\sqrt{2-(-7)}} = 0 \Rightarrow f(g(x))$ is not defined $\therefore$ $x > -7$. But not that $\sqrt{3-\sqrt{2-x}}$ is also undefined when $\sqrt{2-x}$ is undefined, which happens when $2-x < 0$. $\therefore$ $2 - x \ge 0 \Rightarrow 2 \ge x$. All in all, we have $-7 < x \le 2$.

8. What is $g^{-1}(x)$?

   (a) $\sqrt{3 - \sqrt{2 - 1/x}}$

   (b) $-x^2 + 6x - 7$

   (c) $5 - x^2$

   (d) $2 - (x^2 - 3)^2$

   (e) $\dfrac{1}{\sqrt{3 - \sqrt{2-x}}}$

Answer d). Note, for $g(x) = y = \sqrt{3-\sqrt{2-x}}$, our g^{-1} is a function so that $g^{-1}(y) = x$. So, we can go from $g(x) = y$ to g^{-1} by deriving x as a function of y. To that end,

$$y = \sqrt{3-\sqrt{2-x}} \Rightarrow y^2 = 3 - \sqrt{2-x} \Rightarrow y^2 - 3 = -\sqrt{2-x}$$

$$\Rightarrow (y^2-3)^2 = (-\sqrt{2-x})^2 \Rightarrow (y^2-3)^2 = 2 - x \Rightarrow (y^2-3)^2 - 2 = -x$$

$\Rightarrow x = 2 - (y^2-3)^2$. Relabeling y as x, we get $g^{-1}(x) = 2 - (x^2-3)^2$.

9. Let

$$f(x) = \begin{cases} x+2 & \text{if } x < -1, \\ -x & \text{if } -1 < x < 2, \\ 2-2x & \text{if } 2 \le x. \end{cases}$$

Which of the following statements about $f(x)$ is <u>false</u>?

(a) $f(x)$ is continuous on its domain

(b) $f(x)$ is differentiable on its domain

(c) $f(x)$ has a continuous extension at $x = -1$

(d) $\lim_{x\to 2} f(x) = -2$

(e) $f'(0) = -1$

Answer b) $\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{-x+2}{x-2} = \lim_{x\to 2^-} \frac{-(x-2)}{x-2} = -1$

$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{(2-2x)+2}{x-2} = \lim_{x\to 2^+} \frac{-2x+4}{x-2}$

$= \lim_{x\to 2^+} \frac{-2(x-2)}{x-2} = -2.$ The two limits don't agree so it is not differentiable at $x=2$

10. A particle is moving along a straight line. After 2 seconds, it is located at 3.0 m moving at a rate of 3.0 m/s. After 4 seconds, it is located at 5.0 m moving at a rate of 2.0 m/s. Which of the following necessarily happened at some point between 2 and 4 seconds?

(a) The particle was located at 2.0 m

(b) The particle was moving at a rate of 1.0 m/s

(c) The particle was moving at a rate of 0.5 m/s

(d) The particle was accelerating at a rate of 0.5 m/s²

(e) The particle was accelerating at a rate of −1.0 m/s²

Answer b).

Between 2 and 4 seconds the particle has travelled 2 m. Thus, it averaged 1 m/s during that time interval. The mean value theorem tells us that at some time between 2 and 4 seconds it was actually travelling at 1 m/s.

11. What is the range of $f^{-1}(x)$ if $f(x) = \ln(2x - 10)$?

(a) $(5, \infty)$

(b) $(0, 5)$

(c) $(-\infty, \infty)$

(d) $(-\infty, 5) \cup (5, \infty)$

(e) None of the above

Answer a)

If $f(x) = \ln(2x-10)$, then the domain of f is the range of f^{-1}. The domain of $\ln(x)$ is the positive reals, so the domain of $\ln(2x-10)$ is the set of points where $2x-10>0 \Rightarrow 2x>10 \Rightarrow x>5$. So the domain of $\ln(2x-10)$, which is the range of f^{-1}, is $(5,\infty)$

12. Suppose

$$f(x) = \begin{cases} ae^x & \text{if } x > \ln 2, \\ xe^x + a & \text{if } x \leq \ln 2. \end{cases}$$

For which value of a is $f(x)$ continuous?

(a) $\ln 4$

(b) $\ln \sqrt{8}$

(c) $\ln 2$

(d) $-2 \ln 4$

(e) $\ln e^2$

Answer a).

$$\lim_{x\to \ln 2^+} f = \lim_{x\to\ln 2^+} ae^x = ae^{\ln(2)} = 2a$$

$$\lim_{x\to \ln 2^-} f = \lim_{x\to\ln 2^-} xe^x + a = \ln(2)e^{\ln(2)} + a = 2\ln(2) + a$$

$\therefore \lim_{x\to\ln 2} f$ exists if and only if $2a = 2\ln(2) + a \Rightarrow a = 2\ln(2)$.

But $2\ln(2) = \ln(2) + \ln(2) = \ln(2\cdot 2) = \ln(4)$.

$\therefore$ f is continuous when $a = \ln(4)$.

13. Suppose $f(x) = \dfrac{\tan(x)}{x^3} + 3x$. Then $f'(x)$ equals:

(a) $\dfrac{3x^2\tan(x) - x^3\sec^2(x)}{x^6}$

(b) $\dfrac{x^3\tan(x) + 3x^2\sec^2(x)}{x^6} + 3$

(c) $\dfrac{x^3\sec^2(x) - 3x^2\tan(x)}{x^6} + 3$

(d) $\dfrac{\sec^2(x)}{x^3} + 3$

(e) None of the above

Answer c)

Let $f(x) = g(x) + h(x)$, where $g(x) = \dfrac{\tan(x)}{x^3}$ and $h(x) = 3x$.

$f'(x) = g'(x) + h'(x)$. For $h(x) = 3x$, $h'(x) = 3$.

For $g(x) = \dfrac{\tan(x)}{x^3}$ we use the quotient rule to get

$g'(x) = \dfrac{x^3\sec^2(x) - 3x^2\tan x}{x^6}$. $\therefore f'(x) = g'(x) + h'(x) = \dfrac{x^3\sec^2(x) - 3x^2\tan x}{x^6} + 3$

14. Suppose $f(x) = \ln(g(x)\sqrt{x})$ where g is a differentiable function with $g(3) = \sqrt{3}$ and $g'(3) = \sqrt{3}/2$. Then $f'(3)$ equals:

(a) 2/3

(b) $-2/3$

(c) 1/3

(d) $2\sqrt{3}$

(e) None of the above

Answer a).

For $f(x) = \ln(g(x)\sqrt{x})$, we get that $f'(x) = \dfrac{1}{g(x)\sqrt{x}} \cdot \left(g(x)\cdot\dfrac{1}{2\sqrt{x}} + g'(x)\sqrt{x}\right)$

by applying the chain rule and the product rule.

So $f'(3) = \dfrac{1}{g(3)\sqrt{3}}\left(g(3)\cdot\dfrac{1}{2\sqrt{3}} + g'(3)\sqrt{3}\right)$. We have $g(3) = \sqrt{3}$, $g'(3) = \dfrac{\sqrt{3}}{2}$.

$\therefore f'(3) = \dfrac{1}{\sqrt{3}\cdot\sqrt{3}}\left(\dfrac{\sqrt{3}}{2\sqrt{3}} + \dfrac{\sqrt{3}\cdot\sqrt{3}}{2}\right) = \dfrac{1}{3}\left(\dfrac{1}{2} + \dfrac{3}{2}\right) = \dfrac{1}{3}\left(\dfrac{4}{2}\right) = \dfrac{2}{3}$

15. The tangent line to the curve $y^2 = xy + 4$ at $(3, 4)$ is:

(a) $-8 = 4x - 5y$

(b) $3 = -5x + 3y$

(c) $12 = x + 3y$

(d) $-5 = -x + 4y$

(e) None of the above

Answer a)

We have $y^2 = xy + 4$, by implicit differentiation.

we get that $2y'y = y + xy' \Rightarrow 2y'y - xy' = y$

$y'(2y - x) = y \Rightarrow y' = \frac{y}{2y - x}$, So at $(3,4)$

$$y' = \frac{4}{2(4) - 3} = \frac{4}{8 - 3} = \frac{4}{5}.$$

$\therefore$ The line tanget to the curve $y^2 = xy + 4$ at $(3,4)$ is given by $y = \frac{4}{5}x + b$, we need to find b. We know it passes through $(3,4)$, so it satisfies $4 = \frac{4}{5}(3) + b$

$$4 = \frac{12}{5} + b \Rightarrow b = 4 - \frac{12}{5} = \frac{20}{5} - \frac{12}{5} = \frac{8}{5}$$

So our line is $y = \frac{4}{5}x + \frac{8}{5} \Rightarrow y - \frac{4}{5}x = \frac{8}{5}$

$\Rightarrow 5y - 4x = 8 \Rightarrow 4x - 5y = -8$

Long answer questions

16. Let $f(x) = x^2 + x$.

(a) Compute the derivative of $f(x)$ using the limit definition of the derivative. Show all of your work and explain your steps to receive any credit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{((x+h)^2 + (x+h)) - (x^2 + x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{x} + h - \cancel{x^2} - \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 + h}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x + h + 1)}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h + 1 = 2x + 1$$

(b) What is the equation of the tangent line to $f(x)$ at $(3, 12)$?

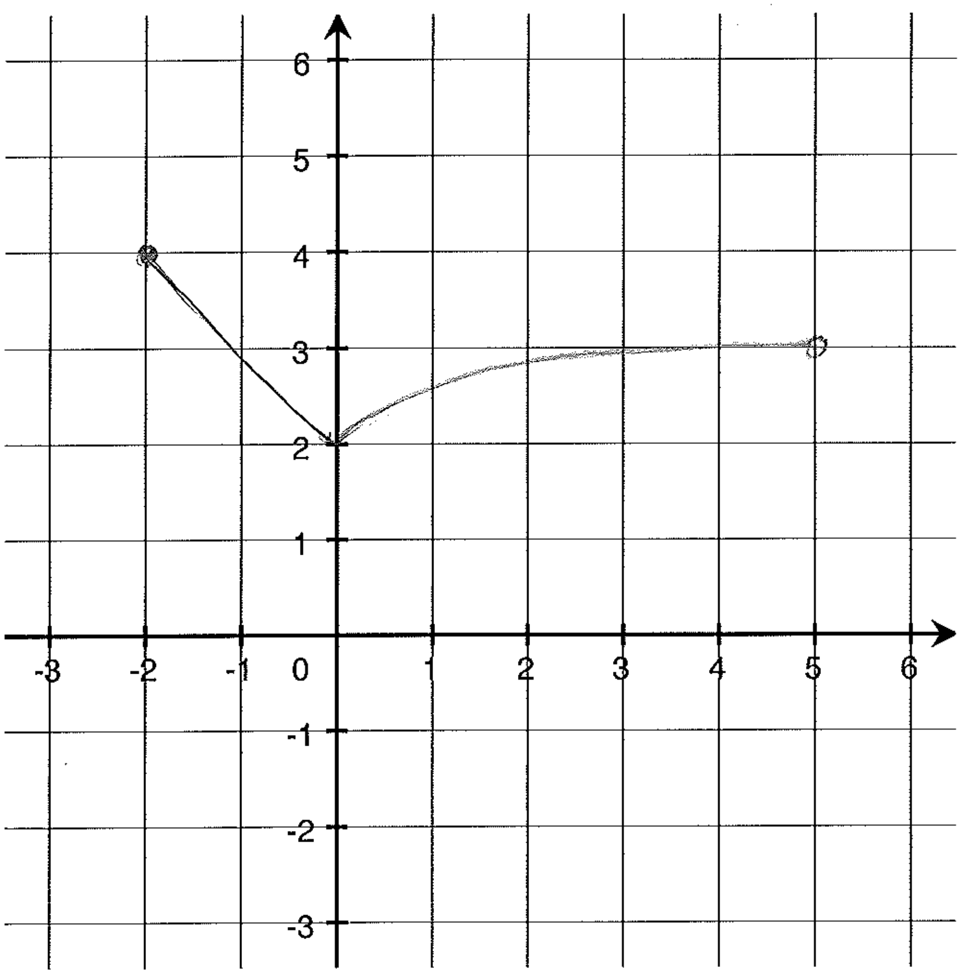
The slope, m, is given by $f'(3) = 2(3) + 1 = 7$, and our line goes through the point $(3, 12)$.

So $12 = 7(3) + b \Rightarrow 12 = 21 + b \Rightarrow b = -9$.

The tangent line at $(3,12)$ is then $y = 7x - 9$

17. Sketch the following function on the axes below.

$$f(x) = \begin{cases} 2 - x & \text{if } -2 \leq x < 0 \\ \sqrt{x+4} & \text{if } 0 \leq x < 5 \end{cases}$$



Is $f(x)$ differentiable at $x = 0$? Choose the best answer.

(a) Yes, because $f(x)$ is continuous at $x = 0$.

(b) Yes, because $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$.

(c) No, because there is a vertical tangent line at $x = 0$.

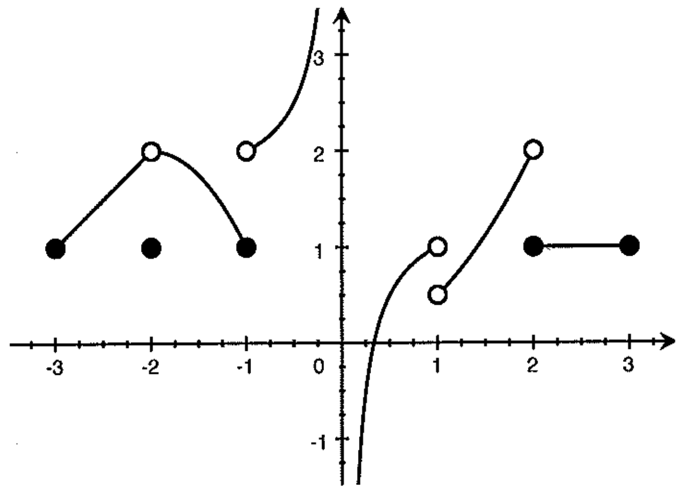
(d) No, because $f(x)$ is a piecewise defined function.

(e) No, because $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist.

Answer e)

$\lim_{x \to 0^+} f(x) = \frac{1}{4}$, $\lim_{x \to 0^-} = -1$, So $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist.

18. Let $f(x)$ be the function graphed below.



(a) For what values of x is $f(x)$ discontinuous? List the x values only.

It is discontinuous at $X = -2$, $X = -1$, $X = 0$, $X = 1$, $X = 2$

(b) Evaluate the following limits:

$$\lim_{x \to -1^-} f(x) = 1$$

$$\lim_{x \to 0^+} f(x) = -\infty$$

$$\lim_{x \to 1^-} ((x-1)^2 f(x)) = 0$$ since $f(x) \to 1$ and $(x-1)^2 \to 0$

$$\lim_{x \to 2^+} (f(x)f(-x)) = 2$$ As $x \to 2^+$ $f(x) \to 1$ and $f(-x) \to 2$

since $x \to 2^+$ of $f(-x)$ is the same as $x \to -2^-$ $f(x)$.

(c) Where does $f(x)$ have removable discontinuities? List the x values for these discontinuities together with the corresponding y values that would remove the discontinuity there.

$f(x)$ has a removable discontinuity at $X = -2$.

If we let $Y = 2$ at $X = -2$, then f would be continuous at $X = -2$.