Math 11 Fall 2007
Solutions to Practice Questions for Exam II

1. Short answer questions.

(a) TRUE or FALSE ?

$$\int_a^b \int_c^d \frac{\partial f}{\partial x}(x,y)\,dy\,dx = f(b,d) - f(a,c)$$

Solution: FALSE, completely false.

(b) Give the best answer:

$$\int\int_R f(x,y)\,dA$$

is guaranteed to exist when R is a closed rectangle in the xy-plane ($a \le x \le b$ and $c \le y \le d$) and the function f

i. is defined at every point of R.
ii. is continuous at every point of R.
iii. is differentiable at every point of R.
iv. None of the above conditions will guarantee the integral exists.

Solution: (ii)

(c) Rewrite the integral with the variables in the opposite order.

$$\int_{-1}^1 \int_{x^2}^1 f(x,y)\,dy\,dx$$

Solution:

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy$$

(d) Rewrite this polar coordinate integral using rectangular coordinates:

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{\cos\theta}} r^2\,dr\,d\theta$$

Solution:

$$\int_0^1 \int_0^x \sqrt{x^2+y^2}\,dy\,dx \quad \text{OR} \quad \int_0^1 \int_y^1 \sqrt{x^2+y^2}\,dx\,dy$$

(e) Find and classify all critical points of the function

$$f(x,y) = x^2 + 8y^2 + 4xy - 4x.$$

Solution:

$$f' = \langle 2x + 4y - 4,\ 16y + 4x \rangle$$

$$D = \begin{vmatrix} 2 & 4 \\ 4 & 16 \end{vmatrix} = 16$$

The only critical point is $\langle 4, -1 \rangle$ and it is a local minimum point.

(f) TRUE or FALSE ?

$$\int_0^1 \int_0^1 \sin(x^2)\sin(y^2)\,dy\,dx = \left[\int_0^1 \sin(x^2)\,dx\right]^2$$

Solution: TRUE

$$\int_0^1 \int_0^1 \sin(x^2)\sin(y^2)\,dy\,dx = \left[\int_0^1 \sin(x^2)\,dx\right]\left[\int_0^1 \sin(y^2)\,dy\right] =$$

$$\left[\int_0^1 \sin(x^2)\,dx\right]\left[\int_0^1 \sin(x^2)\,dx\right] = \left[\int_0^1 \sin(x^2)\,dx\right]^2$$

2. Express

$$\int\int\int_E x\,dV,$$

where E is the region above the xy-plane and below the downward-facing cone $z = 1 - \sqrt{x^2+y^2}$, as an iterated integral in

(a) rectangular

(b) cylindrical

(c) spherical

coordinates. You do not need to evaluate the integral.

Solution:

(a)

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-\sqrt{x^2+y^2}} x\,dz\,dy\,dx$$

(b)

$$\int_0^{2\pi}\int_0^1\int_0^{1-r} r\cos\theta\, r\, dz\, dr\, d\theta$$

(c)

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{1}{\cos\varphi+\sin\varphi}} \rho\cos\theta\sin\varphi\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

3. Find the maximum and minimum values of the function

$$f(x,y) = x^2 + 2y^2 + 3x$$

on the region $x^2 + y^2 \leq 4$.

Solution:

We need to check the values of f at critical points in our region, and on the boundary of the region.

$$f'(x,y) = \langle 2x+3,\ 4y\rangle$$

The only critical point of the function is $\left(-\dfrac{3}{2},\, 0\right)$ and $f\left(-\dfrac{3}{2},\, 0\right) = -\dfrac{9}{4}$. This critical point is in our region $x^2 + y^2 \leq 4$.

To find the largest and smallest values of f on the boundary of the region, $x^2 + y^2 = 4$, we can parametrize the boundary using $(x,y) = (2\cos t,\, 2\sin t)$. Then on the boundary we have

$$f(x,y) = f(2\cos t,\, 2\sin t) = 4\cos^2 t + 8\sin^2 t + 6\cos t =$$

$$4 + 4\sin^2 t + 6\cos t$$

So we need to find the largest and smallest values of

$$g(t) = 4 + 4\sin^2 t + 6\cos t$$

$$g'(t) = 8\sin t\cos t - 6\sin t = \sin t(8\cos t - 6)$$

The largest and smallest values of g will be at its critical points, that is, where

$$\sin t = 0 \quad \text{OR} \quad \cos t = \frac{3}{4}$$

At these points

$$g(t) = 4 \pm 6 \quad \text{OR} \quad g(t) = \frac{41}{4}$$

The largest and smallest values of f on our region are

$$\frac{41}{4} \qquad -\frac{9}{4}$$

The points at which these values are realized are

$$\left(\frac{3}{2}, \pm\frac{\sqrt{7}}{2}\right) \qquad \left(-\frac{3}{2}, 0\right)$$

4. Find the point(s) at which the graph of the function

$$f(x, y) = e^{-x^2-2y^2}$$

is steepest (that is, the point(s) at which the slope of the graph, in the direction of maximal slope, is as large as possible.)

Solution:

To find the (maximum) slope of the graph of f at the point (x, y), we take the magnitude of the gradient of f at that point:

$$\nabla f(x, y) = \left\langle -2xe^{-x^2-2y^2}, -4ye^{-x^2-2y^2} \right\rangle$$

$$|\nabla f(x, y)| = e^{-x^2-2y^2}\sqrt{4x^2 + 16y^2}$$

This is the quantify we have to maximize. To make our calculations easier, we will use a standard trick and instead maximize its square:

$$g(x, y) = (|\nabla f(x, y)|)^2 = e^{-2x^2-4y^2}\left(4x^2 + 16y^2\right)$$

$$\frac{\partial g}{\partial x} = -4xe^{-2x^2-4y^2}\left(4x^2 + 16y^2\right) + e^{-2x^2-4y^2}(8x)$$

$$\frac{\partial g}{\partial y} = -8ye^{-2x^2-4y^2}\left(4x^2 + 16y^2\right) + e^{-2x^2-4y^2}(32y)$$

Critical points of g occur where both partials are zero. Since $e^{-2x^2-4y^2}$ is never zero, we can factor this (as well as the largest constant factor of all terms) out of both partials, and we see we need to solve:

$$x\left(-2x^2 - 8y^2 + 1\right) = 0$$

$$y\left(-x^2-4y^2+1\right)=0$$

The critical points are

$$(0,0) \qquad \left(0,\,\pm\frac{1}{2}\right) \qquad \left(\pm\sqrt{\frac{1}{2}},\,0\right)$$

Going back to $|\nabla f|$, we can compute the maximum slope of the surface at these points, and we get (respectively)

$$0 \qquad e^{-\frac{1}{2}}(2) \qquad e^{-\frac{1}{2}}\sqrt{2}$$

The surface is steepest at the points

$$\left(0,\,\pm\frac{1}{2}\right)$$

5. Find the volume of the region inside the sphere of radius 2 centered at the origin and above the plane $z=1$.

   Solution:

   The sphere and the plane intersect where

   $$x^2+y^2+z^2=4 \qquad z=1,$$

   so in particular we have $x^2+y^2=3$. The volume in question lies above the disc D given by $x^2+y^2\le 3$ in the xy-plane, above the plane $z=1$, and below the top half sphere $z=\sqrt{4-x^2-y^2}$.

   Therefore we can write the volume as

   $$\iint_D \sqrt{4-x^2-y^2}\,dA - \iint_D 1\,dA = \iint_D \sqrt{4-x^2-y^2}\,dA - area(D) =$$

   $$\iint_D \sqrt{4-x^2-y^2}\,dA - 3\pi.$$

   The integral is best set up in polar coordinates.

   $$\iint_D \sqrt{4-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{4-r^2}\,r\,dr\,d\theta = \frac{14\pi}{3}.$$

   Our volume is

   $$\frac{14\pi}{3}-3\pi=\frac{5\pi}{3}.$$

6. Write down a double integral (or a sum of double integrals) representing the volume of the portion of the first octant above the plane $z = 2x+2y$ and below the surface $z = 3 - x^2 - y^2$. Do not evaluate the integral.

Solution:

The surface $z = 2x + 2y$ is a plane through the origin. The surface $z = 3-x^2-y^2$ is a downward-facing paraboloid with top point $(3, 0, 0)$. This second surface looks sort of like a hill, and the plane chops off the top of the hill; we want to find the volume of the portion of the chopped-off top that lies in the first octant.

Our first task is to figure out where these two surfaces intersect. We have

$$z = 2x + 2y = 3 - x^2 - y^2,$$

which gives us

$$\begin{aligned} 2x + 2y &= 3 - x^2 - y^2, \\ x^2 + y^2 + 2x + 2y + 2 &= 5, \\ (x + 1)^2 + (y + 1)^2 &= 5. \end{aligned}$$

This is the equation of a circle of radius $\sqrt{5}$ with center $(-1, -1)$. The portion of the xy-plane lying beneath the three-dimensional region we are considering is the portion of the first quadrant inside this circle. This can be described by the equations

$$0 \le x \le 1 \qquad 0 \le y \le \sqrt{5 - (x + 1)^2} - 1.$$

We want the volume of the region above this portion of the xy-plane, also above the plane $z = 2x+2y$, and below the surface $z = 3-x^2-y^2$. We can find this by finding the volume above this portion of the xy-plane and below the surface $z = 3-x^2-y^2$ and subtracting the volume above this portion of the xy-plane and below the plane $z = 2x + 2y$. This gives us

$$\int_0^1 \int_0^{\sqrt{5-(x+1)^2}-1} 3 - x^2 - y^2 \, dy \, dx - \int_0^1 \int_0^{\sqrt{5-(x+1)^2}-1} 2x + 2y \, dy \, dx.$$

7. A spherical solid of radius 1 centered at the origin has mass density at point P given by

$$1 + (\text{distance from } P \text{ to } z\text{-axis})^2.$$

Find its total mass.

Solution: We need to integrate the mass density function over the region, that is, over the solid sphere of radius 1 around the origin. We can express the mass density in rectangular, cylindrical, or spherical coordinates as

$$1 + x^2 + y^2$$

$$1 + r^2$$

$$1 + \rho^2 \sin^2 \varphi$$

and we can express the boundary of the region (the sphere of radius 1 around the origin) in these coordinate systems as

$$x^2 + y^2 + z^2 = 1$$

$$r^2 + z^2 = 1$$

$$\rho = 1$$

Therefore we can set up the integral in these coordinate systems as

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \left(1 + x^2 + y^2\right)\, dz\, dy\, dx$$

$$\int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} \left(1 + r^2\right)\, r\, dz\, dr\, d\theta$$

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 \left(1 + \rho^2 \sin^2 \varphi\right)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

The first integral is rather intractable, but either of the other two can be evaluated without undue difficulty. The answer is $\dfrac{28\pi}{15}$.